Differential Equations and Special Functions

2.1 Infinite Series

We are mainly interested in those special functions that appear as solutions of differential equations. As suggested in Section 1.3, the solutions can be expressed as infinite series. For this reason, it is convenient here to review some general properties of such series.\(^1\)

In cases of physical interest we require that the series converge. To make clear what we mean by convergence of an infinite series we consider the sum of the first \(n\) terms of the series, which we denote by

\[ S_n = \sum_{k=1}^{n} u_k, \]

and which we call the \(n\)th partial sum.

If the sequence of partial sums converges to a finite limit \(S\) according to

\[ \lim_{n \to \infty} S_n = S, \]

then we say that the infinite series converges and has the value \(S\). That is,

\[ \sum_{k=1}^{\infty} u_k = S. \]

If the terms \(u_k\) vary in sign, then some partial cancellation occurs and convergence is much more likely. If the series

\[ \sum_{k=1}^{\infty} |u_k| \]

converges, then we say that the series

\[ \sum_{k=1}^{\infty} u_k \]

\(^1\)For a more complete discussion see W. Kaplan, *op. cit.* or E.C. Titchmarsh, *op. cit.*
is absolutely convergent. The convergence of a particular series may be established according to one of several convergence tests.\(^2\)

Of more interest to us is an infinite series of functions \(u_k(x)\) of some variable \(x\). Consider the series

\[
S(x) = \sum_{k=1}^{\infty} u_k(x)
\]

for \(x\) in the interval \(a \leq x \leq b\). The series converges if, given any \(x\) in the interval and any positive number \(\varepsilon\), an \(n\) exists such that the absolute difference between the series and the \(n\)th partial sum is less than \(\varepsilon\),

\[
|S(x) - S_n(x)| < \varepsilon \quad \text{with} \quad a \leq x \leq b.
\]

We take \(n\) to be the smallest integer for which this holds.

Notice that in general \(n\) depends on \(\varepsilon\) and on \(x\). It may happen that for each \(\varepsilon\) another number \(N\) exists that is independent of \(x\) such that \(n(x, \varepsilon) < N(\varepsilon)\) for all \(x\) in the interval. In this case we say that the series converges uniformly in the interval.

Infinite series that converge uniformly have some very useful properties, which we now list without proof.\(^3\)

a. The sum of a uniformly convergent series of continuous functions is continuous.

b. A uniformly convergent series of continuous functions can be integrated term by term.

c. A uniformly convergent series of continuous functions can be differentiated term by term if the terms all have continuous derivatives and the series of derivatives converges uniformly.

For the series considered in this book we assume that all of these properties hold unless explicitly stated to the contrary.

### 2.2 Analytic Functions

Generally, the functions we encounter in physics and applied mathematics are analytic. To say that a function \(f(x)\) is analytic for values of the variable \(x\) in the range \(a < x < b\) we mean that for each point \(x_0\) in this interval \(f(x)\) can be written as a power series,

\[
f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n,
\]

\(^2\)See, for example, W. Kaplan, op. cit. or E.C. Titchmarsh, op. cit.

\(^3\)See E.C. Titchmarsh, op. cit.