Non-uniform smoothing

Although the linear scale-space representation generated by smoothing with the rotationally symmetric Gaussian kernel provides a theoretically well-founded framework for handling image structures at different scales, the scale-space smoothing has the negative property that it leads to shape distortions. For example, smoothing across “object boundaries” can affect both the shape and the localization of edges in edge detection. Similarly, surface orientation estimates computed by shape-from-texture algorithms are affected, since the anisotropy of a surface pattern may decrease when smoothed using a rotationally symmetric Gaussian.

15.1. Non-linear diffusion: Review

In order to reduce the shape distortion problems in edge detection, Perona and Malik (1990) proposed the use of anisotropic diffusion. The basic idea is to modify the conductivity $c(x; t)$ in the non-linear diffusion equation

$$\partial_t L = \nabla^T (c(x; t) \nabla L)$$

(15.1)

such as to favour intra-region smoothing to inter-region smoothing. In principle, they solved the diffusion equation

$$\partial_t L = \nabla^T (h(|\nabla L(x; t)|) \nabla L)$$

(15.2)

for some monotonic decreasing function $h: \mathbb{R}_+ \rightarrow \mathbb{R}_+$. The intuitive effect of this evolution is that the conductivity will be low where the gradient magnitude is high and vice versa.

This idea has been further developed by several authors. Nordström (1990) showed that by adding a bias term to the diffusion equation, it was possible to relate this method to earlier considered regularization approaches by Terzopoulos (1983) and Mumford and Shah (1985).

By adopting an axiomatic approach, Alvarez et al. (1992) have shown that given certain constraints on a visual front-end, a natural choice of non-linear diffusion equation is the equation

$$\partial_t L = |\nabla L| \nabla^T (\nabla L / |\nabla L|) = |\nabla L| \kappa(L) = L_{\kappa},$$

(15.3)
where $\kappa(L)$ denotes the curvature of level curves in $L$ (used for junction detection in chapter 6 and chapter 13), and $L_{aa}$ represents the second order derivative in the tangent direction to a level curve. This evolution means that level curves move in the normal direction with a velocity proportional to the curvature of the level curves. If the differential equation (15.3) is slightly modified into

$$\partial_t L = (|\nabla L|^2 \kappa(L))^{1/3} = (\tilde{\kappa}(L))^{1/3},$$

(15.4)

(where $\tilde{\kappa}(L)$ is the rescaled level curve curvature used for junction detection in section 6.2.2 and section 11.3), then it can be shown that its solutions are relative invariant under affine transformations of the spatial coordinates (Alvarez et al. 1992). This property has been used by Sapiro and Tannenbaum (1993) for defining an affine invariant curve evolution scheme.

An interesting approach to describing non-linear diffusion more generally is pursued by Florack et al. (1993), who consider general non-linear coordinate transformations of the spatial coordinates as a means of expressing such operations. Interestingly, this approach covers several of the above mentioned methods.

15.1.1. Properties of non-linear diffusion methods

Trivially, it follows from the maximum principle that if $h > 0$ then any non-linear scale-space representation of the form (15.1) satisfies the causality requirement, or equivalently, the non-enhancement property of local extrema: $\partial_t L < 0$ at local maxima, and $\partial_t L > 0$ at local minima.

The maximum principle does, however, not extend to higher-order derivatives. This can be easily seen in the one-dimensional case

$$\partial_t L = \partial_x (h(|L_x|)L_x) = h_x(|L_x|)L_x + h(|L_x|)L_{xx}. \tag{15.5}$$

Introduce $\phi(L_x) = h(|L_x|)L_x$. Then, (15.5) can be written

$$\partial_t L = \partial_x (\phi(L_x)) = \phi'(L_x)L_{xx}, \tag{15.6}$$

and the evolution of the gradient follows

$$\partial_t L_x = \phi''(L_x)L_{xx}^2 + \phi'(L_x)L_{xxx}. \tag{15.7}$$

For a local gradient maximum with $L_{xx} = 0$ and $L_{xxx} < 0$ it holds that $\partial_t L_x > 0$ if $\phi''(L_x) < 0$. For the conductance function used by Perona and Malik (1990)

$$h(|\nabla L|) = e^{-|\nabla L|^2/k^2}, \tag{15.8}$$