In this chapter, we discuss the time-discretization of time-dependent equations. Although the methods apply to general nonlinear time-dependent equations, their analysis is developed in the linear case and, more especially, for the advection-diffusion equation. First, we address the stability of the spectral approximation, namely, the existence of a bounded solution of the differential equations in time resulting from the spectral approximation. Then the major part of the chapter is devoted to the presentation and discussion of the accuracy and stability of the time-discretization schemes. Second- and higher-order methods are considered in the following two cases: two-step methods (essentially based on Backward-Differentiation and Adams-Bashforth schemes) and one-step methods (Runge-Kutta schemes). The chapter ends with a comparison between the different kinds of time-discretization schemes.

4.1 Introduction

This chapter is devoted to the application of spectral methods to the solution of time-dependent partial differential equations of the form

$$\partial_t u = H(u),$$

(4.1)

where $H$ is a spatial second-order differential operator. With a view to the application to viscous incompressible fluid mechanics, $H$ is the sum of a
nonlinear first-order term \( N(u) \) and a linear second-order term \( L(u) \):

\[
H(u) = N(u) + L(u). \tag{4.2}
\]

In the one-dimensional case, these operators are explicitly

\[
N(u) = -\partial_x F(u), \quad L(u) = \nu \partial_{xx} u, \tag{4.3}
\]

where \( \nu \) is a nonnegative constant. The Burgers equation, already considered in Sections 2.8 and 3.5.2 is obtained with \( F(u) = u^2/2 \). When \( F(u) = a u \) with \( a = \) constant, the resulting equation is the advection-diffusion equation which will be addressed in this chapter. However, the two-dimensional advection-diffusion equation defined by

\[
N(u) = -\mathbf{V} \cdot \nabla u, \quad L(u) = \nu \nabla^2 u, \tag{4.4}
\]

where the vector \( \mathbf{V} \) is given, will also be addressed to introduce some special issues.

The advection-diffusion equation constitutes a convenient model for discussing the numerical methods devised for the solution of the equations governing fluid motion and related transport-diffusion phenomena. For example, considering the motion of an incompressible, viscous, and heat-conducting fluid, the transport-diffusion of momentum \( \rho \mathbf{V} \), vorticity \( \omega \), and temperature \( T \) is governed by an advection-diffusion equation with the fluid velocity \( \mathbf{V} \) as an advective velocity. Obviously, the equation for momentum also involves the pressure gradient.

However, the coupling between the various equations of motion may induce some special properties which are not apparent when the advection-diffusion is considered alone. Such a phenomenon occurs, for example, in the stability of some time-discretization schemes applied to the vorticity-streamfunction equations as will be discussed in Section 6.3.2.e.

The time-discretization is based on finite-difference approximations. The disparity between the finite accuracy of finite-differences used for time-stepping and the "infinite" accuracy reached by the spatial spectral approximation is a question which must be considered. From the conceptual point of view, it would be more satisfactory to have the same kind of accuracy for space and time discretizations. This is the approach chosen by Morchoisne (1979, 1981) who derived a Chebyshev approximation for space and time (see Peyret and Taylor, 1983). However, the modest improvement in accuracy does not justify the cost in computing time and memory of this approach.

Presently, the current spectral codes make use of finite-difference methods for time-discretization. In order to preserve the high accuracy of the spectral method, the truncation error associated with the finite-difference approximation must be sufficiently small. Second-order methods are commonly used, but the relative low order of the scheme is generally counterbalanced by the small value of the time-step required for stability. However,