CHAPTER IX

The Real and Complex Numbers

IX, §1. ORDERING OF RINGS

Let $R$ be an integral ring. By an ordering of $R$ one means a subset $P$ of $R$ satisfying the following conditions:

ORD 1. For every $x \in R$ we have $x \in P$, or $x = 0$, or $-x \in P$, and these three possibilities are mutually exclusive.

ORD 2. If $x, y \in P$ then $x + y \in P$ and $xy \in P$.

We also say that $R$ is ordered by $P$, and call $P$ the set of positive elements.

Let us assume that $R$ is ordered by $P$. Since $1 \neq 0$, and $1 = 1^2 = (-1)^2$ we see that 1 is an element of $P$, i.e. 1 is positive. By ORD 2 and induction, it follows that $1 + \cdots + 1$ (sum taken $n$ times) is positive. An element $x \in R$ such that $x \neq 0$ and $x \notin P$ is called negative. If $x, y$ are negative elements of $R$, then $xy$ is positive (because $-x \in P$, $-y \in P$, and hence $(-x)(-y) = xy \in P$). If $x$ is positive and $y$ is negative, then $xy$ is negative, because $-y$ is positive, and hence $x(-y) = -xy$ is positive.

For any $x \in R$, $x \neq 0$, we see that $x^2$ is positive.

Suppose that $R$ is a field. If $x$ is positive and $x \neq 0$ then $xx^{-1} = 1$, and hence by the preceding remarks, it follows that $x^{-1}$ is also positive.

Let $R$ be an arbitrary ordered integral ring again, and let $R'$ be a subring. Let $P$ be the set of positive elements in $R$, and let $P' = P \cap R$.

Then it is clear that $P'$ defines an ordering on $R'$, which is called the induced ordering.

More generally, let $R'$ and $R$ be ordered rings, and let $P', P$ be their sets of positive elements respectively. Let $f: R' \rightarrow R$ be an embedding
(i.e. an injective homomorphism). We shall say that $f$ is **order-preserving** if for every $x \in R'$ such that $x \in P$ we have $f(x) \in P$. This is equivalent to saying that $f^{-1}(P) = P'$ [where $f^{-1}(P)$ is the set of all $x \in R'$ such that $f(x) \in P$].

Let $x, y \in R$. We define $x < y$ (or $y > x$) to mean that $y - x \in P$. Thus to say that $x > 0$ is equivalent to saying that $x \in P$; and to say that $x < 0$ is equivalent to saying that $x$ is negative, or $-x$ is positive. One verifies easily the usual relations for inequalities, namely for $x, y, z \in R$:

IN 1. $x < y$ and $y < z$ implies $x < z$.

IN 2. $x < y$ and $z > 0$ implies $xz < yz$.

IN 3. $x < y$ implies $x + z < y + z$.

If $R$ is a field, then

IN 4. $x < y$ and $x, y > 0$ implies $1/y < 1/x$.

As an example, we shall prove IN 2. We have $y - x \in P$ and $z \in P$, so that by ORD 2, $(y - x)z \in P$. But $(y - x)z = yz - xz$, so that by definition, $xz < yz$. As another example, to prove IN 4, we multiply the inequality $x < y$ by $x^{-1}$ and $y^{-1}$ to find the assertion of IN 4. The others are left as exercises.

If $x, y \in R$ we define $x \leq y$ to mean that $x < y$ or $x = y$. Then one verifies at once that IN 1, 2, 3 hold if we replace throughout the $<$ sign by $\leq$. Furthermore, one also verifies at once that if $x \leq y$ and $y \leq x$ then $x = y$.

In the next theorem, we see how an ordering on an integral ring can be extended to an ordering of its quotient field.

**Theorem 1.1.** Let $R$ be an integral ring, ordered by $P$. Let $K$ be its quotient field. Let $P_K$ be the set of elements of $K$ which can be written in the form $a/b$ with $a, b \in R$, $b > 0$ and $a > 0$. Then $P_K$ defines an ordering on $K$ extending $P$.

**Proof.** Let $x \in K$, $x \neq 0$. Multiplying a numerator and denominator of $x$ by $-1$ if necessary, we can write $x$ in the form $x = a/b$ with $a, b \in R$ and $b > 0$. If $a > 0$ then $x \in P_K$. If $-a > 0$ then $-x = -a/b \in P_K$. We cannot have both $x$ and $-x \in P_K$, for otherwise, we could write

$$x = a/b \quad \text{and} \quad -x = c/d$$

with $a, b, c, d \in R$ and $a, b, c, d > 0$. Then

$$-a/b = c/d,$$