WEIGHTED $L^q$-THEORY AND POINTWISE ESTIMATES FOR STEADY STOKES AND NAVIER-STOKES EQUATIONS IN DOMAINS WITH EXIT TO INFINITY

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ABSTRACT

In the paper we consider the stationary Stokes and Navier-Stokes equations of a viscous incompressible fluid in domains $\Omega$ with $m > 1$ exits to infinity, which have in some coordinate system the following form

$$\Omega_i = \{x : |x'| < g(x_n), \ x_n > 0\},$$

where $g_i$ are functions satisfying the global Lipschitz condition and $g_i(x_n) \to 0$ as $x_n \to \infty$. We present the theory concerning the solvability of the Stokes and Navier-Stokes systems with prescribed fluxes in weighted Sobolev and Hölder spaces and we show the pointwise decay of the solutions.

INTRODUCTION

The solvability of the boundary and initial–boundary value problems for Stokes and Navier-Stokes equations is one of the most important questions in the mathematical hydrodynamics. It has been studied in many papers and monographs (e.g. Ladyzhenskaya, 1969; Temam, 1977; Galdi, 1994). The existence theory which is developed there concerns mainly the domains with compact boundaries (bounded or exterior). Although some of these results do not depend on the shape of the boundary, many problems of scientific interest concerning the flow of a viscous incompressible fluid in domains with noncompact boundaries were unsolved. Therefore, it is not surprising that during the last 17 years the special attention was given to problems in such domains.

J. Heywood (1976) has shown that in domains with noncompact boundaries the motion of a viscous fluid is not always uniquely determined by the applied external forces and by the usual initial and boundary conditions. Moreover, certain physically
important quantities (as fluxes of the velocity field or limiting values of the pressure at infinity) should be prescribed additionally. For instance, J. Heywood considered the aperture domain

$$\Omega = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \neq 0 \text{ or } x_3 = 0, \ x' = (x_1, x_2) \in S \}, \quad (0.1)$$

where \( S \) is a bounded region in the plane \( \mathbb{R}^2 \). For such domain the Stokes problem

$$\begin{cases}
-\nu \Delta \vec{u} + \nabla p = \vec{f} & \text{in } \Omega, \\
\text{div} \vec{u} = 0 & \text{in } \Omega, \\
\vec{u} = 0 & \text{on } \partial \Omega
\end{cases} \quad (0.2)$$

admits infinitely many solutions with a finite Dirichlet integral. The unique solution of (0.2) can be specified by prescribing either the total flux \( F \) of the fluid through the aperture \( S \), i.e.

$$\int_S u_3(x', x_3) \, dx' = F, \quad (0.3)$$

or the “pressure drop”

$$p_* = p_+ - p_- = \lim_{|x| \to \infty} p(x) - \lim_{|x| \to \infty} p(x) \quad (0.4)$$

The necessity to prescribe additional conditions (0.3), (0.4) is related to the fact that the spaces of divergence free vector fields \( \tilde{H}(\Omega) \) and \( H(\Omega) \) are not identical. For the domain (0.1) one has

$$\dim \tilde{H}(\Omega)/H(\Omega) = 1 \quad (0.5)$$

(see Heywood, 1976). Moreover, in Heywood (1976) the existence of weak solutions to (0.2), (0.3) (or (0.2), (0.4)) has been proved for arbitrary data \( \vec{f}, F \) and \( p_* \). The analogous problems for the nonlinear stationary Navier-Stokes system was solved in Heywood (1976) for small data.

In the subsequent papers of Ladyzhenskaya and Solonnikov (1976, 1977), Solonnikov and Pileckas (1977), Solonnikov (1981, 1983), Kapitanskii (1981), Kapitanskii and Pileckas (1983) it was found that the spaces \( \tilde{H}(\Omega) \) and \( H(\Omega) \) are different for a wide class of domains \( \Omega \subset \mathbb{R}^n, \ n = 2,3 \), having more than one exit to infinity, i.e. the set \( \{ x \in \Omega : |x| > R_0 \} \) is a union of \( m > 1 \) unbounded disjoint domains \( \Omega_1, \ldots, \Omega_m \) which are called “exits to infinity”. The number of additional conditions which should be prescribed for the correct formulation of the Stokes problem is equal to \( \dim \tilde{H}(\Omega)/H(\Omega) \). In turn, the dimension of \( \tilde{H}(\Omega)/H(\Omega) \) is less than \( m \) and depends on the number of exits \( \Omega_i \) blowing up at infinity “not to slowly”. The solution to problem (0.2), having the finite Dirichlet integral, is completely determined by prescribing exactly \( \dim \tilde{H}(\Omega)/H(\Omega) \) additional conditions in “wide” exits to infinity. The weak solvability of stationary Stokes and Navier-Stokes problems with prescribed additional side conditions in “wide” exits to infinity was proved for arbitrary data in Ladyzhenskaya and Solonnikov (1976, 1977), Solonnikov and Pileckas (1977), Solonnikov (1981, 1983), Kapitanskii and Pileckas (1983). Notice that the domain (0.1) has two “wide” exits to infinity and, therefore, in Ladyzhenskaya and Solonnikov (1977), Solonnikov and Pileckas (1977), Solonnikov (1981), Kapitanskii and Pileckas (1983), the results of Heywood (1976), concerning the nonlinear problem, are extended to arbitrary large data.

\(^1\tilde{H}(\Omega) \) is the completion in the Dirichlet norm of all divergence free vector functions with compact supports in \( \Omega \) and \( \tilde{H}(\Omega) \) is the space of all divergence free vector functions having finite Dirichlet norm and zero traces on \( \partial \Omega \).