§16. DUAL INTEGRAL EQUATIONS

16.1. The Electrified Disc

To motivate this section, we first solve a classical problem of electrostatics. We wish to find the electrostatic potential $\phi$ created by an isolated thin conducting disc of radius $a$, whose potential is $V$. Noting the symmetry of the problem about the axis of the disc and introducing cylindrical polar coordinates $r, \theta, \text{and} z$, we reduce the problem to that of satisfying the equations

$$\phi_{rr} + \frac{1}{r} \phi_r + \phi_{zz} = 0$$

(1)

and

$$\phi(r,0) = V, \quad r < a,$$
$$\phi_z(r,0^+) = \phi_z(r,0^-), \quad r > a.$$  

(2)

Applying the Hankel transform of order zero, we easily find from (1) that

$$\phi(k,z) = A(k)e^{-k|z|},$$

(3)

and that the boundary conditions (2) reduce to the "dual integral equations"

$$\int_0^\infty A(k) J_0(kr)kd k = V, \quad r < a,$$  

(4)

$$\int_0^\infty kA(k) J_0(kr)kd k = 0, \quad r > a.$$

If we differentiate (4a) with respect to $r$, we obtain an alternative pair of equations, namely

$$\phi_r(r,0) = -\int_0^\infty A(k) J_1(kr)k^2dk$$
$$= 0, \quad r < a,$$

(5)
§16. Dual integral equations

\[ \phi_z(r,0^+) = \int_0^\infty A(k) J_0(kr) k^2 \, dk \]
\[ = 0, \ r > a. \tag{6} \]

From (15.21) we see that the function
\[ A(k) = C(ka)^{-3/2} J_{1/2}(ka) \tag{7} \]
satisfies both of these equations; furthermore, with this form for \( A(k) \), (15.21) gives
\[ \phi_r(r,0) = -\frac{C}{a^2\sqrt{\frac{k}{\pi}}} \frac{h(r-a)}{r} \frac{h(r-a)}{r} \]
and thus
\[ \phi(r,0) = -\int_0^\infty \phi_t(t,0) \, dt \]
\[ = \begin{cases} \frac{C}{a^2\sqrt{\frac{k}{\pi}}} \sin^{-1}(a/r), & r > a \\ \frac{C}{a^2\sqrt{\frac{k}{\pi}}} & r < a. \end{cases} \tag{9} \]
Finally, this implies \( C = Va^2\sqrt{2/\pi} \), so the solution is
\[ \phi(r,z) = \frac{2V}{\pi} \int_0^\infty \sin(ka)e^{-k|z|} k \, J_0(kr) \, dk \ . \tag{10} \]

16.2. Dual Integral Equations of Titchmarsh Type

Equations of the type
\[ \int_0^\infty k^{-2\alpha} A(k) J_{\mu}(kx) \, dk = f(x), \ x < a, \tag{11} \]
\[ \int_0^\infty k^{-2\beta} A(k) J_{\nu}(kx) \, dk = g(x), \ x > a, \]
where \( f(x) \) and \( g(x) \) are only known over part of the range \( 0 < x < \infty \) and \( A(k) \) is sought, occur in certain mixed boundary value problems of which the electrified disc