Chapter VI

The Wave Equation

I. THE ONE-DIMENSIONAL WAVE EQUATION

Consider the hyperbolic equation

\[ u_{tt} - c^2 u_{xx} = 0. \tag{1.1} \]

The variable \( t \) stands for time, and one-dimensional refers to the number of space variables. A general solution of (1.1) in a convex domain \( \Omega \) of \( \mathbb{R}^2 \) is given by

\[ u(x, t) = F(x - ct) + G(x + ct), \tag{1.2} \]

where \( s \rightarrow F(s), G(s) \) are of class \( C^2 \) within their domain of definition. Indeed, the change of variables

\[ \xi = x - ct, \quad \eta = x + ct \tag{1.3} \]

transforms \( \Omega \) into a convex domain \( \tilde{\Omega} \) of the \( (\xi, \eta) \)-plane and in terms of \( \xi \) and \( \eta \), (1.1) becomes

\[ U_{\xi \eta} = 0; \quad U(\xi, \eta) = u \left( \frac{\xi + \eta}{2}, \frac{\eta - \xi}{2c} \right). \tag{1.4} \]

Therefore, \( U_\xi = F'(\xi) \) and

\[ U(\xi, \eta) = \int F'(\xi) \, d\xi + G(\eta). \]

Rotating the axes back of an angle

\[ \theta = \arctan \left( \frac{1}{c} \right) \cap (0, \frac{\pi}{2}), \]

maps \( \tilde{\Omega} \) into \( \Omega \) in the \( (x, t) \)-plane and

\[ u(x, t) = F(x - ct) + G(x + ct). \]

Remark 1.1. The graphs of \( \xi \rightarrow F(\xi) \) and \( \eta \rightarrow G(\eta) \) are called undistorted waves propagating to the right and left respectively (right and left here refer to the positive orientation of the \( x \), and \( t \), axes). The two lines obtained from (1.3) by making \( \xi \) and \( \eta \) constants are called characteristic lines. Writing them in the parametric form

\[ \begin{cases} x_1(t) = ct + \xi, & t \in \mathbb{R}, \\ x_2(t) = -ct + \eta, \end{cases} \]

we may regard the abscissas \( t \rightarrow x_i(t), \ i = 1, 2, \) as points travelling on the \( x \)-axis, with velocities \( \pm c \) respectively.
1.1. A Property of Solutions

Consider any parallelogram of vertices $A$, $B$, $C$, $D$ with sides parallel to the characteristics $x = \pm ct + \xi$ and contained in some convex domain $G \subset \mathbb{R}^2$.

![Figure 1.1](image)

We call it a characteristic parallelogram. Let

- $A \equiv (x, t)$,
- $B \equiv (x + cs, t + s)$
- $C \equiv (x + cs - c\tau, t + s + \tau)$
- $D \equiv (x - c\tau, t + \tau)$

be the coordinates of the vertices of a characteristic parallelogram, where $s$ and $\tau$ are positive parameters. If $u$ is of class $C^2$ in $G$ and solves (1.1), it follows from (1.2) that

\[
\begin{align*}
u(A) &= F(x - ct) + G(x + ct) \\
u(C) &= F(x - 2c\tau - ct) + G(x + 2cs + ct) \\
u(B) &= F(x - ct) + G(x + 2cs + ct) \\
u(D) &= F(x - 2c\tau - ct) + G(x + ct).
\end{align*}
\]

Therefore,

\begin{equation}
(1.5) \quad u(A) + u(C) = u(B) + u(D).
\end{equation}

Every solution of the form (1.2) satisfies (1.5). Vice versa if $u$ is of class $C^2$ in $G$ and satisfies (1.5) for any characteristic parallelogram, we rewrite (1.5) as

\[
[u(x, t) - u(x + cs, t + s)] - [u(x - ct, t + \tau) - u(x + cs - c\tau, t + s + \tau)] = 0.
\]

Using the Taylor formula we see that $u$ satisfies the p.d.e. (1.1). We may regard (1.5) as a weak formulation of (1.1).\(^1\)

\(^1\) See Problem 1.2 of the Complements.