12. THE OPTIMAL CONTROL OF AN EXCITABLE NEURAL FIBRE

J.E. Rubio and A.V. Holden

Departments of Mathematics and Physiology
University of Leeds
Leeds, U.K.

We consider the optimal control of a neural fibre, described by a nonlinear diffusion equation with a polynomial nonlinearity. An iterative scheme is established to compute a minimum-energy control, in which at each step of the iteration a linear problem is solved by means of measure-theoretical methods and linear programming. The method converges for moderate values of the nonlinear terms. Some numerical results are given.

1. Introduction

We develop in this paper a procedure for the design of an optimal control for a nonlinear diffusion equation which describes the behaviour of a neural fibre. This equation has been much studied \([1,2,6]\), but no serious mathematical attempt seems to have been made to develop a useful method for the design of optimal controls for it.

The nonlinearity will be considered to have a polynomial form; if \(y\) is the potential of the membrane, the nonlinear term will be taken as \(f(y) = y[y(a-y)+z]\), where \(a\) and \(z\) are numerical parameters much discussed in the literature, especially by Casten et al. [1]. This function has been used by some authors to model other phenomena such as combustion or the spread of epidemics [6].

Our approach is a direct extension of some of our work on the control of the linear diffusion equation \([4,5]\). After choosing a performance criterion, an iterative scheme is established so that at each iteration a linear problem is solved, by means of an approach involving measure theory and linear programming. The scheme converges for moderate values of the constant \(\gamma\) which multiplies the nonlinearity \(f\), and a good approximation to the optimal control for the nonlinear equation is thus obtained.

2. The problem

Consider a nonlinear diffusion equation

\[
y_{xx}(x,t) = y_t(x,t) + \gamma f(y(x,t)), \quad (x,t) \in (0,1) \times (0,T)
\]  

(1)
where \( y \) is the variable representing potential difference, \( \gamma \) is a numerical parameter, \( T \) is the final value of the time, and \( f: \mathbb{R} \to \mathbb{R} \) is a nonlinear polynomial function of the form

\[
f(y) = y(a-y)(1-y)+z
\]

where \( a \) and \( z \) are numerical parameters to be chosen below.

The boundary conditions are:

\[
\begin{align*}
\frac{\partial y}{\partial x}(0,t) &= 0, \quad t \in [0,T], \\
\frac{\partial y}{\partial x}(1,t) &= u(t), \quad t \in [0,T], \\
y(x,0) &= 0, \quad x \in [0,1];
\end{align*}
\]

(3)

where \( u(t), t \in [0,T], \) is the control, which can be interpreted as the current being injected into the fibre at the end defined by \( x=1; \) the other end is insulated. The control \( u \) will be termed admissible if it is measurable, \( u(t) \in [-1,1] \) ae and \( y(x,T) = g(x) \), a given function which is the desired final state, on \( [0,1] \). That is, we put a (normalized) amplitude constraint on the input current, and we wish to reach a given state \( g \) at the final time \( T \). For instance, it could be that we wish to stop a possible wild behaviour of the fibre by applying a control that will bring this to a stop at the final time \( T \). The set of all admissible controls, to be denoted by \( U \), is assumed to be nonempty.

Let \( f^0 \in \text{Lips}(\Omega) \), the space of Lipschitz-continuous functions on \( \Omega = [0,T] \times [-1,1] \). The control problem consists in finding a control \( u \in U \) so as to minimize the functional

\[
J(u) = \int_0^T f^0(t,u(t))\,dt;
\]

(4)

that is, we are interested in reaching the final state \( g \) by means of a control with values in \([-1,1]\), while at the same time minimizing the integral criterion (4), which may itself represent the total energy used in the process - if the integrand is \( u^2 \) - or the total charge - if the integrand is \( |u| \).

3. Linear problems and measures

We shall develop an algorithm to solve this problem for values of the parameter \( \gamma \in [0,\gamma_0] \), for some \( \gamma_0 \in \mathbb{R} \); this will be an iterative procedure, in which a sequence of related linear problems are solved. Firstly - before introducing iteration - we shall consider the linear problem associated with that above. Let \( F \in C((0,1) \times (0,T)) \), and let

\[
\frac{\partial^2 y}{\partial x^2}(x,t) = \frac{\partial y}{\partial t}(x,t) + F(x,t), \quad (x,t) \in (0,1) \times (0,T),
\]

(5)

with the same boundary conditions as in (2). This is a linear partial differential equation, with an explicit solution given by:

\[
y(x,T) = \sum_{n=0}^{\infty} a_n \cos(n\pi x),
\]

(6)