8

MATRICES AND DETERMINANTS

8.1. Introduction

Consider a family with parents named $P_1$ and $P_2$ who have very imaginatively named their two children $C_1$ and $C_2$. Let us say each month the children get an allowance, also denoted by the symbols $C_1$ and $C_2$ which are related to the parents' income (likewise denoted by $P_1$ and $P_2$) as follows:

\[
\begin{align*}
C_1 &= \frac{1}{10}P_1 + \frac{1}{6}P_2 \\
C_2 &= \frac{1}{9}P_1 + \frac{1}{8}P_2.
\end{align*}
\]

In other words the father gives $\frac{1}{10}$ of his income to the son and the mother gives $\frac{1}{6}$ of hers to the son. The daughter similarly receives $\frac{1}{9}$ and $\frac{1}{8}$ respectively from her father and mother.

Let us assume that no matter what the incomes of the parents, they will always contribute these fractions. In that case we would like to store the unchanging fractions $\frac{1}{10}, \frac{1}{6}, \frac{1}{9}, \frac{1}{8}$ in some form. It is logical to store them in an array or matrix

\[
M = \begin{bmatrix}
\frac{1}{10} & \frac{1}{6} \\
\frac{1}{9} & \frac{1}{8}
\end{bmatrix}
\]

The logic is as follows. The contributions naturally fall into two sets: what the son gets and what the daughter gets. This makes the two rows. In each row, there are again two sets: what the father gives and what the mother gives. These make the two columns. In the general case we can write the parental contributions as

\[
M = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\]

The matrix element $M_{ij}$ resides in row $i$ and column $j$ and stands for the contribution to child $i$ from parent $j$. We can then write in the general case

\[
\begin{align*}
C_1 &= M_{11}P_1 + M_{12}P_2 \\
C_2 &= M_{21}P_1 + M_{22}P_2.
\end{align*}
\]
Chapter 8

Let us generalize this notion of a matrix to an array with \( m \) rows and \( n \) columns. A matrix with \( m \) rows and \( n \) columns will be referred to as an \( m \) by \( n \) or \( m \times n \) matrix. Let us introduce two 2 by 1 matrices:

\[
C = \begin{bmatrix}
C_1 \\
C_2
\end{bmatrix}
\]  

(8.1.7)

and

\[
P = \begin{bmatrix}
P_1 \\
P_2
\end{bmatrix}
\]  

(8.1.8)

A matrix with \( m = n \) is called a square matrix, a matrix with \( n = 1 \) is called a column vector, and a matrix with \( m = 1 \) is called a row vector. We will be dealing only with these three types of matrices. In addition, with a few exceptions, the name matrix will be used to mean a square matrix, while column and row vectors will be referred to as such.

We next introduce the notion of adding and multiplying matrices. In the world of pure mathematics one can make up rules as one wants, as long as they are consistent. However some definitions prove to be more useful than others, as is true of the ones that follow.

**Definition 8.1.** If \( M \) and \( N \) are two matrices of the same dimensions (same number of rows and columns) their sum \( T = M + N \) has entries \( T_{ij} = M_{ij} + N_{ij} \).

**Definition 8.2.** If \( a \) is a number, \( aM \) is defined as a matrix with entries \( (aM)_{ij} = aM_{ij} \).

**Definition 8.3.** If \( M \) is an \( m \) by \( n \) matrix and \( N \) is an \( n \) by \( r \) matrix, their product \( MN \) is an \( m \) by \( r \) matrix with entries

\[
(MN)_{ij} = \sum_{r=1}^{n} M_{ir}N_{rj}
\]

(8.1.9)

Whereas the sum is easy to remember, the product takes some practice.

To get the \( ij \) matrix element of the product you must take the entries of the \( i \)-th row of the first matrix, interpret them as components of a vector, and take the dot product with the \( j \)-th column of the second matrix, similarly interpreted.

The product is not defined if the number of columns of the first matrix do not equal the number of rows of the second. Likewise the sum is defined only for matrices of the same dimension. Here are a few examples.

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\begin{bmatrix}
4 & 2 \\
1 & 3
\end{bmatrix}
= \begin{bmatrix}
6 & 8 \\
16 & 18
\end{bmatrix}
\]

(8.1.10)