Discrete probability distributions

6.1 INTRODUCTION

If a discrete variable can take values with associated probabilities it is called a discrete random variable. The values and the probabilities are said to form a discrete probability distribution.

For example, the discrete probability distribution for the variable number of heads resulting from tossing a fair coin three times may be represented as in Table 6.1.

<table>
<thead>
<tr>
<th>Number of heads</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.125</td>
</tr>
<tr>
<td>1</td>
<td>0.375</td>
</tr>
<tr>
<td>2</td>
<td>0.375</td>
</tr>
<tr>
<td>3</td>
<td>0.125</td>
</tr>
</tbody>
</table>

The above probabilities may be determined by the use of a probability tree (see Worksheet 5, Question 15).

There are several standard types of discrete probability distribution. We will consider two of the most important, namely the binomial distribution and the Poisson distribution.

6.2 BINOMIAL DISTRIBUTION, AN EXAMPLE

Consider another coin-tossing example, but this time we will toss the coin 10 times. The number of heads will vary if we repeatedly toss the coin 10 times and we may note the following:

1. We have a fixed number of tosses, that is 10.
2. Each toss can result in one of only two outcomes, heads and tails.
3. The probability of heads is the same for all tosses, and is $\frac{1}{2}$ for a fair coin.
4. The tosses are independent in the sense that the probability of heads for any toss is unaffected by the result of previous tosses.
Because these four conditions are satisfied, the experiment of tossing the coin 10 times is an example of what is called a binomial experiment (consisting of 10 so-called ‘Bernoulli’ trials).

The variable ‘number of heads in the 10 tosses of a coin’ is said to have a binomial distribution with ‘parameters’ 10 and 0.5, which we write in a short-hand form as \(B(10, 0.5)\). The first parameter, 10, is the number of trials or tosses and the second parameter, 0.5, is the probability of heads in a single trial or toss.

### 6.3 THE GENERAL BINOMIAL DISTRIBUTION

To generalize the example of the previous section, the outcomes of each trial in a binomial experiment are conventionally referred to as ‘success’ (one of the outcomes) and ‘failure’. The general binomial distribution is denoted by \(B(n, p)\) where the parameters \(n\) and \(p\) are the number of trials and the probability of success in a single trial, respectively.

In order to decide a priori whether a variable has a binomial distribution we must check the four conditions (generalizing on those of the previous section):

1. There must be a fixed number of trials, \(n\).
2. Each trial can result in one of only two outcomes, which we refer to as success and failure.
3. The probability of success in a single trial, \(p\), is constant.
4. The trials are independent, so that the probability of success in any trial is unaffected by the results of previous trials.

If all four conditions are satisfied then the discrete random variable, which we call \(x\), to stand for the number of successes in \(n\) trials, has a \(B(n, p)\) distribution.

Unless \(n\) is small (\(\leq 3\), say) the methods we used in Chapter 5 are inefficient for calculating probabilities. Luckily we can use a formula, which we shall quote without proof, or in some cases we can use tables, see Section 6.6, to find probabilities for a particular binomial distribution, if we know the numerical values of the parameters, \(n\) and \(p\).

The formula is

\[
P(x) = \binom{n}{x} p^x (1 - p)^{n-x}
\]

for \(x = 0, 1, 2, \ldots, n\).

This formula is not difficult to use if each part is understood separately: \(P(x)\) means ‘the probability of \(x\) successes in \(n\) trials’.

\(\binom{n}{x}\) is a shorthand for \(n!/x!(n-x)!\), where \(n!\) means ‘factorial \(n\’ (refer to Section 2.2 if necessary).

\(x = 0, 1, 2, \ldots, n\) means that we can use this formula for each of these values of \(x\), which are the possible numbers of successes in \(n\) trials.