7.1 Steady flow in a two-dimensional channel

In previous chapters, we derived the differential equations governing the motion of an incompressible Newtonian fluid by requiring mass conservation and enforcing Newton’s second law of motion for infinitesimal fluid parcels. The governing equations are accompanied by initial, boundary, and interfacial conditions, as required. In this chapter, we proceed to derive analytical and semi-analytical solutions for an important class of steady and unsteady flows with rectilinear or circular streamlines. The engineering significance of these flows, combined with their ability to demonstrate the salient mechanisms of momentum and vorticity transport at steady or unsteady state, justify a extensive consideration.

7.1 Steady flow in a two-dimensional channel

We begin by considering flow in a two-dimensional channel confined between two parallel plane walls that are separated by distance $2a$ and are inclined by an angle $\beta$ with respect to the horizontal plane, as illustrated in Figure 7.1.1.

In the inclined system of Cartesian coordinates defined in Figure 7.1.1, the $x$ axis is parallel to the walls and the $y$ axis is perpendicular to the walls. The corresponding Cartesian components of the acceleration of gravity vector are

$$
g_x = g \sin \beta, \quad g_y = -g \cos \beta, \quad (7.1.1)
$$

where $g$ is the magnitude of the acceleration of gravity. The lower wall translates parallel to itself with constant velocity $V_1$, and the upper wall translates parallel to itself with constant velocity $V_2$. 

The motion of the fluid is governed by the Navier–Stokes equation (6.5.6) whose Cartesian components are displayed in Table 6.5.1.

**Unidirectional and fully developed flow**

Our analysis will be based on the assumption of steady unidirectional flow, requiring that the $y$ and $z$ velocity components vanish, $u_y = 0$ and $u_z = 0$, while the $x$ component remains constant in time, $\partial u_x / \partial t = 0$. This assumption precludes the occurrence of turbulent motion where small-scale three-dimensional fluctuations are observed, as discussed in Chapter 10. The continuity equation for two-dimensional flow,

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0,$$

requires that $\partial u_x / \partial x = 0$, which states that the flow is fully developed. Thus, the axial velocity, $u_x$, is a function of position across the channel, $y$, alone, that is, $u_x(y)$.

**Governing equations and pressure field**

Simplifying the $x$ and $y$ components of the equation of motion shown in Table 6.5.1 by discarding terms that are identically zero, we obtain

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{d^2 u_x}{dy^2} + \rho g_x$$

and

$$0 = -\frac{\partial p}{\partial y} + \rho g_y.$$ 

In fact, equation (7.1.4) determines the pressure distribution in hydrostatics.