1. Local Decompositions of Control Systems

1.1 Introduction

The subject of this Chapter is the analysis of a nonlinear control system, from the point of view of the interaction between input and state and – respectively – between state and output, with the aim of establishing a number of interesting analogies with some fundamental features of linear control systems. For convenience, and in order to set up an appropriate basis for the discussion of these analogies, we begin by reviewing – perhaps in a slightly unusual perspective – a few basic facts about the theory of linear systems.

Recall that a linear multivariable control system with \( m \) inputs and \( p \) outputs is usually described, in state space form, by means of a set of first order linear differential equations

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\] (1.1)

in which \( x \) denotes the state vector (an element of \( \mathbb{R}^n \)), \( u \) the input vector (an element of \( \mathbb{R}^m \)) and \( y \) the output vector (an element of \( \mathbb{R}^p \)). The matrices \( A, B, C \) are matrices of real numbers, of proper dimensions.

The analysis of the interaction between input and state, on one hand, and between state and output, on the other hand, has proved of fundamental importance in understanding the possibility of solving a large number of relevant control problems, including eigenvalues assignment via feedback, minimization of quadratic cost criteria, disturbance rejection, asymptotic output regulation, etc. Key tools for the analysis of such interactions – introduced by Kalman around the 1960 – are the notions of reachability and observability and the corresponding decompositions of the control system into “reachable/unreachable” and, respectively, “observable/unobservable” parts. We review in this section some relevant aspects of these decompositions.

Consider the linear system (1.1), and suppose that there exists a \( d \)-subspace \( V \) of \( \mathbb{R}^n \) having the following property:

(i) \( V \) is invariant under \( A \), i.e. is such that \( Ax \in V \) for all \( x \in V \).

Without loss of generality, after possibly a change of coordinates, we can assume that the subspace \( V \) is the set of vectors having the form \( v = \text{col}(v_1, \ldots, v_d, 0, \ldots, 0) \), i.e. of all vectors whose last \( n - d \) components
are zero. If this is the case, then, because of the invariance of $V$ under $A$, this matrix assumes necessarily a block triangular structure

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

with zero entries on the lower-left block of $n - d$ rows and $d$ columns.

Moreover, if the subspace $V$ is such that:

(ii) $V$ contains the image (i.e. the range-space) of the matrix $B$, i.e. is such that $Bu \in V$ for all $u \in \mathbb{R}^m$,

then, after the same change of coordinates, the matrix $B$ assumes the form

$$B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

i.e. has zero entries on the last $n - d$ rows.

Thus, if there exists a subspace $V$ which satisfies (i) and (ii), after a change of coordinates in the state space, the first equation of (1.1) can be decomposed in the form

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u$$
$$\dot{x}_2 = A_{22}x_2.$$ 

By $x_1$ and $x_2$ we denote here the vectors formed by taking the first $d$ and, respectively, the last $n - d$ new coordinates of a point $x$.

The representation thus obtained is particularly interesting when studying the behavior of the system under the action of the control $u$. At any time $T$, the coordinates of $x(T)$ are

$$x_1(T) = \exp(A_{11}T)x_1(0) + \int_0^T \exp(A_{11}(T - \tau))A_{12}\exp(A_{22}\tau)d\tau x_2(0) +$$
$$+ \int_0^T \exp(A_{11}(T - \tau))B_1u(\tau)d\tau$$
$$x_2(T) = \exp(A_{22}T)x_2(0).$$

From this, we see that the set of coordinates denoted by $x_2$ does not depend on the input $u$ but only on the time $T$. In particular, if we denote by $x^\circ(T)$ the point of $\mathbb{R}^n$ reached at time $t = T$ when $u(t) = 0$ for all $t \in [0, T]$, i.e. the point

$$x^\circ(T) = \exp(AT)x(0)$$

we observe that any state which can be reached at time $T$, starting from $x(0)$ at time $t = 0$, has necessarily the form $x^\circ(T) + v$, where $v$ is an element of $V$.

This argument identifies only a necessary condition for a state $x$ to be reachable at time $T$, i.e. that of being of the form $x = x^\circ(T) + v$, with $v \in V$. However, under the additional assumption that: