2. Global Decompositions of Control Systems

2.1 Sussmann’s Theorem and Global Decompositions

In the previous Chapter, we have shown that a nonsingular and involutive distribution \( \Delta \) induces a local partition of the state space into lower dimensional submanifolds and we have used this result to obtain local decompositions of control systems. The decompositions thus obtained are very useful to understand the behavior of control systems from the point of view of input-state and, respectively, state-output interaction. However, it must be stressed that the existence of decompositions of this type is strictly related to the assumption that the dimension of the distribution is constant at least over a neighborhood of the point around which we want to investigate the behavior of our control system.

In this section we shall see that the assumption that \( \Delta \) is nonsingular can be removed and that global partitions of the state space can be obtained. Since we are interested in establishing results which have a global validity, it is convenient – for more generality – to consider, as anticipated in section 1.5, the case of control systems whose state space is a manifold \( N \). Of course, this more general analysis will cover in particular the case in which \( N = U \).

To begin with, we need to introduce a few more concepts. Let \( \Delta \) be a distribution defined on the manifold \( N \). A submanifold \( S \) of \( N \) is said to be an integral submanifold of the distribution \( \Delta \) if, for every \( p \in S \), the tangent space \( T_pS \) to \( S \) at \( p \) coincides with the subspace \( \Delta(p) \) of \( T_pN \). A maximal integral submanifold of \( \Delta \) is a connected integral submanifold \( S \) of \( \Delta \) with the property that every other connected integral submanifold of \( \Delta \) which contains \( S \) coincides with \( S \). We see immediately from this definition that any two maximal integral submanifolds of \( \Delta \) passing through a point \( p \in N \) must coincide. Motivated by this, it is said that a distribution \( \Delta \) on \( N \) has the maximal integral manifolds property if through every point \( p \in N \) passes a maximal integral submanifold of \( \Delta \) or, in other words, if there exists a partition of \( N \) into maximal integral submanifolds of \( \Delta \).

It is easily seen that this is a global version of the notion of complete integrability for a distribution. As a matter of fact, a nonsingular and completely integrable distribution is such that for each \( p \in N \) there exists a neighborhood \( U \) of \( p \) with the property that \( \Delta \) restricted to \( U \) has the maximal integral manifolds property.
A simple consequence of the previous definitions is the following one.

**Lemma 2.1.1.** A distribution $\Delta$ which has the maximal integral manifolds property is involutive.

**Proof.** If $\tau$ is a vector field which belongs to a distribution $\Delta$ with the maximal integral manifolds property, then $\tau$ must be tangent to every maximal integral submanifold $S$ of $\Delta$. As a consequence, the Lie bracket $[\tau_1, \tau_2]$ of two vector fields $\tau_1$ and $\tau_2$ both belonging to $\Delta$ must be tangent to every maximal integral submanifold $S$ of $\Delta$. Thus, $[\tau_1, \tau_2]$ belongs to $\Delta$. \(\triangleright\)

Thus, involutivity is a necessary condition for $\Delta$ to have the maximal integral manifolds property but, if $\Delta$ has points of singularity, this condition may fail to be sufficient.

**Example 2.1.1.** Let $N = \mathbb{R}^2$ and let $\Delta$ be a distribution defined by

$$\Delta(x) = \text{span}\left\{ \left( \frac{\partial}{\partial x_1} \right)_x \lambda(x_1), \left( \frac{\partial}{\partial x_2} \right)_x \right\}$$

where $\lambda(x_1)$ is a $C^\infty$ function such that $\lambda(x_1) = 0$ for $x_1 \leq 0$ and $\lambda(x_1) > 0$ for $x_1 > 0$. This distribution is involutive and

\[
\begin{align*}
\dim \Delta(x) &= 1 & \text{if } x \text{ is such that } x_1 \leq 0 \\
\dim \Delta(x) &= 2 & \text{if } x \text{ is such that } x_1 > 0.
\end{align*}
\]

Clearly, the open subset of $N$

$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$$

is an integral submanifold of $\Delta$ (actually a maximal integral submanifold) and so is any subset of the form

$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0, x_2 = c\}.$$

However, it is not possible to find integral submanifolds of $\Delta$ passing through a point $(0, c)$. \(\triangleright\)

Another important point to be stressed, which emphasizes the difference between the general problem considered here and its local version described in section 1.4, is that the elements of a global partition of $N$ induced by a distribution which has the integral manifolds property are immersed submanifolds. On the contrary, local partitions induced by a nonsingular and completely integrable distribution are always made of slices of a coordinate neighborhood, i.e. of embedded submanifolds.