When Is a Linear Continuous-time System Easy or Hard to Control in Practice?

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Summary. This paper is focused on characterization of easily controllable plants in practical control applications rather than to design an optimal or a robust controller for a given plant. After explaining the background and the motivation of the research topic, we first provide two notions, namely finite frequency positive realness (FFPR) and Condition (π), which represent desirable phase/gain properties for easily controllable plants. We then show closed-form analytical expressions of best achievable H₂ tracking and regulation performances, and we provide the connection between Condition (π) and the achievable robust performance based on H∞ loop shaping design procedure.

Keywords: Feedback control, Control performance limitation, Phase/gain property, H₂ control, H∞ loop shaping design, Reciprocal transform.

1 Introduction

More robust and high performance is required for feedback control in a lot of applications of advanced technology. It is necessary to design a good plant as well as an optimal controller, since there exist control performance limitations caused by plant properties such as unstable poles, non-minimum phase zeros, lightly damped modes, and time delays.

This paper focuses on characterization of easily controllable plants in practical control applications rather than to design an optimal or a robust controller for a given plant. In other words, we consider a plant design to guarantee existence of a controller that achieves desirable closed-loop performance. Once we design a plant with such a property, standard optimal and/or robust control methods can be applied to complete the whole design process. The key point is that the control performance is explicitly taken into account in the process of plant design.

The well-known Bode integral type relations tell us that unstable or non-minimum phase plants are not easy to control. However, it is not clear what kind of properties are really required for plants to be controlled in order to achieve the desirable feedback performance under physical constraints such as control effort limit and measurement accuracy. In this paper, we will make a partial answer to the question by summarizing several recent results by the authors, which relate researches on control performance limitations [13, 14].
The paper is organized as follows. Section 2 is devoted to motivation and background of the research, which indicates that the notion of minimum phase is not enough for our purpose. Two notions based on the phase/gain property, finite frequency positive realness (FFPR) [7] and Condition (π) [6,8] are introduced in Section 3 for characterizing a set of easily controllable plants in practice. Sections 4 and 5 are devoted to \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) control performance limitations. Two types of \( \mathcal{H}_2 \) tracking and regulation performance limits are provided in Section 4. Section 5 investigates the connection between Condition (π) and the achievable robust performance. Some concluding remarks are made in Section 6.

**Notation:** \( \mathbb{R} \) and \( \mathbb{C} \) respectively denote the sets of real and complex numbers, and we define \( \mathbb{C}_+ := \{ s \in \mathbb{C} \mid \text{Re}(s) > 0 \} \) and \( \mathbb{C}_- := \{ s \in \mathbb{C} \mid \text{Re}(s) < 0 \} \).

## 2 Intrinsic Control Performance Limits

### 2.1 Bode Integral Relations

Consider the typical SISO (single-input and single-output) unity feedback control system depicted in Fig. 1, where \( P(s) \) denotes the continuous-time plant to be controlled and \( K(s) \) is the continuous-time controller to be designed. \( r(t), d(t), u(t), y(t), \) and \( e(t) := r(t) - y(t) \) are the reference command, disturbance input, control input, plant output, and error signal, respectively.

The sensitivity function \( S(s) \) and the complementary sensitivity function \( T(s) \) respectively defined by

\[
S(s) := \frac{1}{1 + L(s)}, \quad T(s) := \frac{L(s)}{1 + L(s)}
\]

play an important role for evaluating the control performance, where \( L(s) := P(s)K(s) \) is the loop transfer function. The most well-known result on control performance limitation is the so called Bode integral relation on sensitivity gain which is given as follows [3, 13]:

Suppose that \( L(s) \) is strictly proper, i.e., \( S(\infty) = 1 \), and that the closed-loop system is stable. Then, we have

\[
\frac{1}{\pi} \int_{0}^{\infty} \log |S(j\omega)| \, d\omega = \sum_{k=1}^{n_p} p_k^a - \frac{1}{2} \nu_\infty; \quad \nu_\infty := \lim_{s \to \infty} s \cdot L(s), \tag{1}
\]

where \( p_k^a \in \mathbb{C}_+ (k = 1, \ldots, n_p) \) are unstable poles of \( L(s) \).

![Fig. 1. Unity feedback control system](image)