In this chapter we give a descriptive account of surfaces, of which we have already met the plane, the sphere and the torus. There are many other surfaces, shortly to be described. The essential idea is that near each of its points a surface is just like the plane.

**Definition 4.1**

A Euclidean set $S$ is a *surface* if each of its points has a neighbourhood homeomorphic to an open disc.

A set consisting of two intersecting cylinders is not a surface: no point of intersection has a neighbourhood of the required form. For the disc, cylinder and Möbius band to be surfaces, we must leave off the edge points.

A closed cylinder, that is, a cylinder with its edge points but without its ends filled in to make a sphere, is not a surface but a surface with boundary. A Euclidean set $S$ is a *surface with boundary* if every point of $S$ has a neighbourhood homeomorphic either to an open disc or to the set $\{(x, y) : x^2 + y^2 < 1, x \geq 0\}$, shown in Figure 4.1. Closed discs, closed cylinders and closed Möbius bands are not surfaces, but are surfaces with boundary.

Our object in this chapter is to describe all those surfaces, like the sphere and the torus, that are bounded, closed and path-connected. To prove that we shall have listed all such surfaces—let us call them *good* surfaces—is beyond, but not far beyond, the scope of this book.
The sphere and the torus are the first of a sequence of good surfaces constructed by adding handles to, or equivalently making holes in, the sphere (see Note 4.1). The torus is the sphere with one hole or handle: the two sets shown in Figure 4.2 are homeomorphic. The double torus has two holes, or two handles. More generally, the sphere with \( k \) holes in, or with \( k \) handles is the surface of genus \( k \). Each of these surfaces is orientable, a property that we describe only informally. To be orientable is to be two-sided. For example, the torus could be painted red on the outside and green on the inside, and is two-sided, as is the surface of genus \( k \) for each \( k \). The Möbius band, though, is non-orientable, or one-sided, for if you start painting a Möbius band red, you end up painting it all red. What gives spice to the study of surfaces is that there are good surfaces other than spheres with handles. These new surfaces are considerably