4. Flows in Networks

The purpose of this chapter is to describe basic elements of the theory and applications of network flows. This topic is probably the most important single tool for applications of digraphs and perhaps even of graphs as a whole. At the same time, from a theoretical point of view, flow problems constitute a beautiful common generalization of shortest path problems and problems such as finding internally (arc)-disjoint paths from a given vertex to another. The theory of flows is well understood and fairly simple. This, combined with the enormous applicability to real-life problems, makes flows a very attractive topic to study. From a theoretical point of view, flows are well understood as far as the basic questions, such as finding a maximum flow from a given source to a given sink or characterizing the size of such a flow, are concerned. However, the topic is still a very active research field and there are challenging open problems such as deciding whether an $O(nm)$ algorithm$^1$ exists for the general maximum flow problem.

Several books deal almost exclusively with flows; see, e.g., the books [13] by Ahuja, Magnanti and Orlin, [267] by Dolan and Aldous, the classical text [331] by Ford and Fulkerson and [710] by Murty. In particular, [13] and [710] contain a wealth of applications of flows. In this chapter we can only cover a very small part of the theory and applications of network flows, but we will try to illustrate the diversity of the topic and show several applications of a practical as well as theoretical nature. Many of the results given in this chapter will be used in several other chapters such as those on connectivity and hamiltonian cycles.

4.1 Definitions and Basic Properties

A network is a directed graph $D = (V, A)$ associated with the following functions on $V \times V$: a lower bound $l_{ij} \geq 0$, a capacity $u_{ij} \geq l_{ij}$ and a cost $c_{ij}$ for each $(i, j) \in V \times V$. These parameters satisfy the following requirement:

$^1$ Here and everywhere in this chapter $n$ is the number of vertices and $m$ the number of arcs in the network under consideration.
For every \((i, j) \in V \times V\), if \(ij \not\in A\), then \(l_{ij} = u_{ij} = 0\). \hfill (4.1)

In order to simplify notation in this chapter we also make the assumption that

\[ c_{ij} = -c_{ji} \quad \forall (i, j) \in V \times V. \hfill (4.2) \]

This assumption may seem restrictive but it is purely a technical convention to make some of the following definitions simpler (in particular, the definition of costs in the residual network in Subsection 4.1.2). When it comes to implementing algorithms for various flow problems involving costs, this assumption can easily be avoided (Exercise 4.2). Finally we assume that if there is no arc between \(i\) and \(j\) (in any direction), then \(c_{ij} = 0\).

In some cases we also have a function \(b : V \rightarrow \mathbb{R}\) called a balance vector which associates a real number with each vertex of \(D\). We will always assume that

\[ \sum_{v \in V} b(v) = 0. \hfill (4.3) \]

We use the shorthand notation \(\mathcal{N} = (V, A, l, u, b, c)\) to denote a network with corresponding digraph \(D = (V, A)\) and parameters \(l, u, b, c\). If there are no costs specified, or there is no prescribed balance vector, then we omit the relevant letters from the notation. Note that whenever we consider a network \(\mathcal{N} = (V, A, l, u, b, c)\) we also have a digraph, namely, the digraph \(D = (V, A)\) that we obtain from \(\mathcal{N}\) by omitting all the functions \(l, u, b, c\).

For a given pair of not necessarily disjoint subsets \(U, W\) of the vertex set of a network \(\mathcal{N} = (V, A, l, u)\) and a function \(f\) on \(V \times V\) we use the notation \(f(U, W)\) as follows (here \(f_{ij}\) denotes the value of \(f\) on the pair \((i, j)\)):

\[ f(U, W) = \sum_{i \in U, j \in W} f_{ij}. \hfill (4.4) \]

We will always make the realistic assumption that \(n = O(m)\) which holds for all interesting networks. In fact, almost always, the networks on which we work will be connected as digraphs.

### 4.1.1 Flows and Their Balance Vectors

A flow in a network \(\mathcal{N}\) is a function \(x : A \rightarrow \mathbb{R}_0^+\) on the arc set of \(\mathcal{N}\). We denote the value of \(x\) on the arc \(ij\) by \(x_{ij}\). For convenience, we will sometimes think of \(x\) as a function of \(V \times V\) and require that \(x_{ij} = 0\) if \(ij \not\in A\) (see, e.g., the definition of residual capacity in (4.7)). An integer flow in \(\mathcal{N}\) is a flow \(x\) such that \(x_{ij} \in \mathbb{Z}_0^+\) for every arc \(ij\). For a given flow \(x\) in \(\mathcal{N}\) the balance vector of \(x\) is the following function \(b_x\) on the vertices: