Chapter 20
The Discrete Fourier Transform

This chapter is a mathematically sophisticated treatment of the theory of Fourier analysis. It concentrates on the discrete Fourier transform which is the variant used in image analysis practice. It is not necessary to know this material to understand the developments in this book; this is meant as supplementary material.

20.1 Linear Shift-Invariant Systems

Let us consider a system $\mathcal{H}$ operating on one-dimensional input signals $I(x)$. The system is linear if for inputs $I_1(x)$ and $I_2(x)$, and scalar $\alpha$

$$
\mathcal{H}\{I_1(x) + I_2(x)\} = \mathcal{H}\{I_1(x)\} + \mathcal{H}\{I_2(x)\},
$$

(20.1)

$$
\mathcal{H}\{\alpha I_1(x)\} = \alpha \mathcal{H}\{I_1(x)\};
$$

(20.2)

similar definitions apply in the two-dimensional case. A system $\mathcal{H}$ is shift-invariant if a shift in the input results in a shift of the same size in the output; that is, if $\mathcal{H}\{I(x)\} = O(x)$, then for any integer $m$

$$
\mathcal{H}\{I(x + m)\} = O(x + m);
$$

(20.3)

or, in the two-dimensional case, for any integers $m$ and $n$,

$$
\mathcal{H}\{I(x + m, y + n)\} = O(x + m, y + n).
$$

(20.4)

A linear shift-invariant system $\mathcal{H}$ operating on signals (or, in the two-dimensional case, on images) can be implemented by either linear filtering with a filter, or another operation, the convolution of the input and the impulse response of the system. The impulse response $H(x)$ is the response of the system to an impulse

$$
\delta(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise}, \end{cases}
$$

(20.5)

that is

$$
H(x) = \mathcal{H}\{\delta(x)\}.
$$

(20.6)

By noting that $I(x) = \sum_{k=-\infty}^{\infty} I(k) \delta(x - k)$, and by applying linearity and shift-invariance properties (equations (20.1)–(20.3)) it is easy to show that

$$
O(x) = \mathcal{H}\{I(x)\} = \sum_{k=-\infty}^{\infty} I(k) H(x - k) = I(x) * H(x),
$$

(20.7)
where the last equality sign defines convolution ∗. Note that convolution is a symmetric operator since by making the change in summation index ℓ = x − k (implying k = x − ℓ)

\[ I(x) ∗ H(x) = \sum_{k=-\infty}^{\infty} I(k)H(x - k) = \sum_{\ell=-\infty}^{\infty} H(\ell)I(x - \ell) = H(x) ∗ I(x). \] (20.8)

### 20.2 One-Dimensional Discrete Fourier Transform

#### 20.2.1 Euler’s Formula

For purposes of mathematical convenience, in Fourier analysis, the frequency representation is complex-valued: both the basis images and the weights consist of complex numbers; this is called the representation of an image in the Fourier space. The fundamental reason for this is Euler’s formula, which states that

\[ e^{ai} = \cos a + i \sin a \] (20.9)

where \( i \) is the imaginary unit. Thus, a complex exponential contains both the sin and cos function in a way that turns out to be algebraically very convenient. One of the basic reasons for this is that the absolute value of a complex number contains the sum-of-squares operation:

\[ |a + bi| = \sqrt{a^2 + b^2} \] (20.10)

which is related to the formula in (2.16) on page 40 which gives the power of a sinusoidal component. We will see below that we can indeed compute the Fourier power as the absolute value (modulus) of some complex numbers.

In fact, we will see that the argument of a complex number on the complex plane is related to the phase in signal processing. The argument of a complex number \( c \) is a real number \( \phi \in (-\pi, \pi] \) such that

\[ c = |c|e^{\phi i}. \] (20.11)

We will use here the signal-processing notation \( \angle c \) for the argument.

We will also use the complex conjugate of a complex number \( c = a + bi \), denoted by \( \bar{c} \), which can be obtained either as \( a - bi \) or, equivalently, as \( |c|e^{-\phi i} \). Thus, the complex conjugate has the same absolute value, but opposite argument (“phase”).

#### 20.2.2 Representation in Complex Exponentials

In signal processing theory, sinusoidals are usually represented in the form of the following complex exponential signal

\[ e^{i\omega x} = \cos(\omega x) + i \sin(\omega x), \quad x = 1, \ldots, M. \] (20.12)