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The Gauss–Bonnet theorem

The Gauss–Bonnet theorem is the most beautiful and profound result in the theory of surfaces. Its most important version relates the average of the Gaussian curvature to a property of the surface called its ‘Euler number’ which is ‘topological’, i.e., it is unchanged by any diffeomorphism of the surface. Such diffeomorphisms will in general change the value of the Gaussian curvature, but the theorem says that its average over the surface does not change. The real importance of the Gauss–Bonnet theorem is as a prototype of analogous results which apply in higher dimensional situations, and which relate geometrical properties to topological ones. The study of such relations was one of the most important themes of twentieth century mathematics, and continues to be actively studied today.

13.1 Gauss–Bonnet for simple closed curves

The simplest version of the Gauss–Bonnet Theorem involves simple closed curves on a surface. In the special case when the surface is a plane, these curves have been discussed in Section 3.1. For a general surface, we make the following definition.
Definition 13.1.1

A curve $\gamma(t) = \sigma(u(t), v(t))$ on a surface patch $\sigma : U \to \mathbb{R}^3$ is called a simple closed curve with period $T$ if $\pi(t) = (u(t), v(t))$ is a simple closed curve in $\mathbb{R}^2$ with period $T$ such that the region $\text{int}(\pi)$ of $\mathbb{R}^2$ enclosed by $\pi$ is entirely contained in $U$ (see the diagrams above). The curve $\gamma$ is said to be positively-oriented if $\pi$ is positively-oriented. Finally, the image of $\text{int}(\pi)$ under the map $\sigma$ is defined to be the interior $\text{int}(\gamma)$ of $\gamma$.

We can now state the first version of the Gauss–Bonnet Theorem.

Theorem 13.1.2

Let $\gamma(s)$ be a unit-speed simple closed curve on a surface patch $\sigma$ of length $\ell(\gamma)$, and assume that $\gamma$ is positively-oriented. Then,

$$\int_0^{\ell(\gamma)} \kappa_g ds = 2\pi - \int_{\text{int}(\gamma)} K dA_\sigma,$$

where $\kappa_g$ is the geodesic curvature of $\gamma$, $K$ is the Gaussian curvature of $\sigma$ and $dA_\sigma$ is the area element of $\sigma$ (see Section 6.4).

We use $s$ to denote the parameter of $\gamma$ to emphasize that $\gamma$ is unit-speed. The double integral on the right-hand side of the equation in Theorem 13.1.2 is called the total curvature of the region $\text{int}(\gamma)$.

Proof

We start by computing the geodesic curvature of $\gamma$. For this, we shall make use of a smooth orthonormal basis $\{e', e''\}$ of the tangent plane at each point of the surface patch, where ‘smooth’ means that $e'$ and $e''$ are smooth functions of