Chapter 4
General Spectral Theory of Nonlinear Systems

4.1 Introduction

The spectral theory of linear (time-invariant) systems is, of course, the most important aspect of control theory in classical feedback design, and historically this was the approach taken by control engineers until the introduction of state-space theory. The techniques developed in the past include Nyquist and Bode diagrams, pole assignment and root locus methods. Frequency domain methods are therefore extremely important, especially for the suppression of resonant vibrations in mechanical systems.

In this chapter we shall outline a general spectral theory for nonlinear systems. First, a generalised transform theory is developed which can generate Volterra series kernels directly using Schwartz’ kernel theorem, without the need of a (somewhat arbitrary) definition of multi-dimensional Laplace transforms. This theory can then be directly applied, by using the iteration scheme developed in this book to derive a general spectral theory for nonlinear systems. We shall then briefly show how to apply the same ideas to generalise exponential dichotomies and the Sacker-Sell spectrum to nonlinear systems. We shall assume a basic knowledge of functional analysis and distribution theory.

4.2 A Frequency-domain Theory of Nonlinear Systems

In this section we outline the methods of [1] and the resulting frequency-domain theory of nonlinear systems. The systems which we consider initially are the bilinear ones of the form

\[ \dot{x} = Ax + uDx + bu, \quad x(0) = x_0. \]

These systems are usually studied by means for the Volterra series (see, e.g. [2]), a typical term of which is of the form
where the kernel \( K_k \) has a number of different representations. In order to make this look like a \( k \)-dimensional convolution, a number of subtle transformations are made and then \( k \)-dimensional Laplace transforms are taken (see [2]). In this section we shall use a direct way of obtaining these results which are valid for a much more general class of inputs.

We begin by defining a very general frequency domain theory for nonlinear systems as given in [3]. Let \( L^2_T[0, \infty) \) be the Hilbert space of all measurable, square-integrable, real-valued functions defined on \([0, \infty)\) and which are zero for \( t > T \). This is clearly a direct subspace of \( L^2[0, \infty) \). Let \( S \) be a nonlinear causal system, i.e. \( S \) maps \( L^2_T[0, \infty) \) to itself for all \( T > 0 \). (For input-output stable systems, we can take \( T = \infty \), i.e. \([0, \infty)\).) Let \( \mathfrak{F} \) denote the Fourier transform, so that \( \mathfrak{F} \) is an isomorphism \( \mathfrak{F}: L^2(-\infty, \infty) \rightarrow L^2(-\infty, \infty) \)

and \( \mathfrak{F} \) maps \( L^2_T[0, \infty) \) one-to-one and isometrically onto a subspace \( \tilde{L}^2_T[0, \infty) \) of \( L^2(-\infty, \infty) \). We define the transformed system \( \tilde{S} \) by

\[
\tilde{S}(v) = \mathfrak{F} S \mathfrak{F}^{-1}(v).
\]

Thus, \( \tilde{S} \) makes the diagram

\[
\begin{array}{ccc}
L^2_T[0, \infty) & \xrightarrow{S} & L^2_T[0, \infty) \\
\mathfrak{F} & \downarrow & \mathfrak{F} \\
\tilde{L}^2_T[0, \infty) & \xrightarrow{\tilde{S}} & \tilde{L}^2_T[0, \infty)
\end{array}
\]

commute. Note that if \( S \) is an analytic function on \( \tilde{L}^2_T[0, \infty) \) (i.e. has a convergent Taylor series consisting of \( S \) and its Fréchet derivatives), then \( \tilde{S} \) is analytic on \( L^2_T[0, \infty) \), since \( \mathfrak{F} \) is linear and invertible. Thus we can expand \( \tilde{S} \) in a Taylor series:

\[
\tilde{S} = \sum_{i=0}^{\infty} M_i(v)
\]

where \( M_i \) is an \( i \) form defined on \( \tilde{L}^2_T[0, \infty) \), i.e. \( M_i = L_i(v, \cdots, v) \) for some multi-linear form \( L_i : \bigoplus_{j=1}^{i} \tilde{L}^2_T[0, \infty) \).

Example 4.1. Consider the linear system given by

\[
S : y(t) = \int_{0}^{t} g(t - \tau) u(\tau) d\tau.
\]