Chapter 4
Constrained Optimization

Abstract In previous chapters, we only searched the space defined by variables’ upper and lower bounds. But real-world problems are always with constraints. One important question that needs to be answered when applying EAs in constrained optimization is how to evaluate a solution that violates some constraints. Generally, we want the final results of our EAs to satisfy all the constraints. But discarding the those that violate some constraints and generating new ones again is very inefficient. Several wonderful ideas for constraint handling will be discussed in this chapter.

4.1 Introduction

4.1.1 Constrained Optimization

The real world is constrained by various rules, so that the mathematical models representing real-world optimization and learning problems will have various constraints. The general form of constrained optimization problems (COPs) can be illustrated as follows:

\[ \begin{align*}
\min & \quad f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n \\
\text{s.t.} & \quad g_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, q \\
& \quad h_j(\mathbf{x}) = 0, \quad j = q + 1, \ldots, k \\
& \quad L_l \leq x_l \leq U_l, \quad l = 1, \ldots, n
\end{align*} \] (4.1)

where \( L_l \) and \( U_l \) are the lower and upper bounds of variable \( x_l \), respectively, which forms the search space \( S \).\(^2\) \( q \) inequality constraints (linear or nonlinear) and \( k - q \) equality constraints (linear or nonlinear) need to be satisfied, which forms the search space \( S \).

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1 If objective or constraint function is nonlinear with \( \mathbf{x} \), it is also called nonlinear programming (NLP).
2 Generally, these \( n \) inequalities are not regarded as constraints but form the definition domain.
feasible region $F$. $F \subseteq S$. If point $x \in F$, we say $x$ is feasible, else if point $x \in S \setminus F$, we say $x$ is infeasible.\footnote{If $x \notin S$, we say $x$ is illegal.} If the purpose of OR is to find feasible solutions, i.e., the objective is not considered, then such problems are called constrained satisfaction problems (CSPs).

If a point $x$ satisfies $g_i(x) = 0$ for inequality constraint $i$, we say that constraint $i$ is active at point $x$. All equality constraints are considered to be active at feasible region $F$.

Due to the possible complex form of constraints, the relationship between $F$ and $S$ might be complicated. Figure 4.1 is one example.

![Feasible region and search space](image)

**Fig. 4.1** Feasible region and search space

In Fig.4.1, $F$ is a nonconvex disconnected set and the feasible optimal solution is at the edge of the feasible region, which is rather far away from the optimal solution without constraints. All of these characteristics of $F$ add difficulties to COPs.

Due to the above facets and considering the No Free Lunch theorem, generally it is impossible to develop a deterministic method for COPs that might be better than an exhaustive search, i.e., a COP is an intractable model for OR methods.

Equality constraints $h_j(x) = 0$ in Eq. 4.1 might be the most difficult part of NLP because they make $F$ extremely small compared to $S$. So generally, for almost all NLP solvers, we need to relax the equality constraints to inequality constraints as follows:

$$|h_j(x)| \leq \delta, \quad j = q + 1, \cdots, k \tag{4.2}$$

where $\delta$ is the tolerance value predefined by users.\footnote{$\delta = 0.001$ or $\delta = 0.0001$ are commonly suggested fixed values. Techniques to control $\delta$ in the evolving process will be introduced later.} In this way, we can transfer Eq. 4.1 into an NLP with $k$ inequality constraints.