Maximum determinant positive definite Toeplitz completions

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Dedicated to our friend Rien Kaashoek in celebration of his eightieth birthday.

Abstract. We consider partial symmetric Toeplitz matrices where a positive definite completion exists. We characterize those patterns where the maximum determinant completion is itself Toeplitz. We then extend these results with positive definite replaced by positive semidefinite, and maximum determinant replaced by maximum rank. These results are used to determine the singularity degree of a family of semidefinite optimization problems.

Mathematics Subject Classification (2010). 15A60, 15A83, 15B05, 90C22.

Keywords. Matrix completion, Toeplitz matrix, positive definite completion, maximum determinant, singularity degree.

1. Introduction

In this paper we study the positive definite completion of a *partial symmetric Toeplitz matrix,* $\mathcal{T}$. The main contribution is Theorem 1.1, where we present a characterization of those Toeplitz patterns for which the maximum determinant completion is Toeplitz, whenever the partial matrix is positive definite completable. Part of this result answers a conjecture about the existence of a positive definite Toeplitz completion with a specific pattern. A consequence of the main result is an extension to the maximum rank completion in the positive semidefinite case, and an application to the *singularity degree* of a family of *semidefinite programs (SDPs)*. In the following paragraphs we introduce relevant background information, state the main result, and motivate our pursuit.

Research of S. Sremac and H. Wolkowicz supported by The Natural Sciences and Engineering Research Council of Canada.
Research of H.J. Woerdeman supported by Simons Foundation grant 355645.

H. Bart et al. (eds.), *Operator Theory, Analysis and the State Space Approach*, Operator Theory: Advances and Applications 271, https://doi.org/10.1007/978-3-030-04269-1_17

A *partial matrix* is a matrix in which some of the entries are assigned values while others are unspecified, treated as variables. For instance,

$$\mathcal{M} := \begin{bmatrix} 6 & 1 & x & 1 & 1 \\ 1 & 6 & 1 & y & 1 \\ u & 1 & 6 & 1 & z \\ 1 & v & 1 & 6 & 1 \\ 1 & 1 & w & 1 & 6 \end{bmatrix} \tag{1.1}$$

is a real partial matrix, where the unspecified entries are indicated by letters. A *completion* of a partial matrix $\mathcal{T}$ is obtained by assigning values to the unspecified entries. In other words, a matrix T (completely specified) is a completion of $\mathcal{T}$ if it coincides with $\mathcal{T}$ over the specified entries: $T_{ij} = \mathcal{T}_{ij}$, whenever $\mathcal{T}_{ij}$ is specified. A *matrix completion problem* is to determine whether the partial matrix can be completed so as to satisfy a desired property. This type of problem has enjoyed considerable attention in the literature due to applications in numerous areas; see, e.g., [2, 28]. For example, matrix completion is used in sensor network localization [23, 24], where the property is that the completion is a Euclidean distance matrix with a given embedding dimension. Related references for matrix completion problems are, e.g., [1, 8, 14, 17, 18].

The pattern of a partial matrix is the set of specified entries. For example, the pattern of $\mathcal{M}$ is all of the elements in diagonals $-4, -3, -1, 0, 1, 3, 4$. Whether a partial matrix is completable to some property may depend on the values assigned to the specified entries (the data) and it may also depend on the pattern of specified entries. A question pursued throughout the literature is whether there exist patterns admitting certain completions whenever the data satisfy some assumptions. Consider, for instance, the property of positive definiteness. A necessary condition for a partial matrix to have a positive definite completion is that all completely specified principal submatrices are positive definite. We refer to such partial matrices as *partially positive definite*. Now we ask: what are the patterns for which a positive definite completion exists whenever a partial matrix having the pattern is partially positive definite? In [13] the set of such patterns is shown to be fully characterized by *chordality of the graph* determined by the pattern.

In this work the desired property is *symmetric Toeplitz positive definite*. In particular, we consider the completion with maximum determinant over all (including non-Toeplitz) positive definite completions. Recall that a real symmetric $n \times n$ matrix T is Toeplitz if there exist real numbers $t_0, \dots, t_{n-1}$ such that $T_{ij} = t_{|i-j|}$ for all $i, j \in \{1, \dots, n\}$. A partial matrix is said to be *partially symmetric Toeplitz* if the specified entries are symmetric and consist of entire diagonals where the data is symmetric and constant over each diagonal. The pattern of such a matrix indicates which diagonals are known and hence is a subset of $\{0, \dots, n-1\}$. Here 0 refers to the main diagonal, 1 refers to the superdiagonal and so on. The subdiagonals need not be specified in the pattern since they are implied by symmetry. In fact, since positive definite completions automatically exist when the main diagonal is