Chapter 12
An Explicit Finite Volume Numerical Scheme for 2D Elastic Wave Propagation

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Abstract The construction of a two-dimensional finite volume numerical scheme based on the representation of computational cells as thermodynamic systems is presented explicitly. The main advantage of the scheme is an accurate implementation of conditions at interfaces and boundaries. It is demonstrated that boundary conditions influence the wave motion even in the simple case of a homogeneous waveguide.

12.1 Introduction

Problems in wave propagation in elastic solids can be formulated in terms of hyperbolic conservation laws. Due to the great importance of conservation laws (Dafermos, 2010), numerous numerical methods have been applied to their solution: finite difference methods (Godlewski and Raviart, 1996; Trangenstein, 2009), finite element methods (Cohen, 2002; Kampanis et al., 2008), discontinuous Galerkin methods (Hesthaven and Warburton, 2007; Cohen and Pernet, 2017), finite volume methods (LeVeque, 2002; Guinot, 2003), spectral methods (Hesthaven et al., 2007; Gopalakrishnan et al., 2007) etc.

A comprehensive survey of numerical methods for conservation laws has been presented recently (Hesthaven, 2018). Nevertheless, problems still exist with interface and boundary conditions in multidimensional cases (Gao et al., 2015). In this paper we focus on the construction of a two-dimensional (2D) explicit finite volume numerical scheme with the special attention to the implementation of boundary conditions.

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12.1.1 Finite Volume Methods

Finite volume schemes are powerful numerical methods for solving nonlinear conservation laws and related equations. Such methods are locally conservative and based on cell averages. The numerical solution of systems of hyperbolic conservation laws is dominated by Riemann-solver-based schemes (Godlewski and Raviart, 1996; Toro, 1997; LeVeque, 2002; Guinot, 2003). An upgrade of the solution in a given cell is determined by the exchanges (via fluxes) at the interfaces with the neighbouring cells. The fluxes are computed by solving Riemann problems at the interfaces between neighbouring cells.

Computing an exact solution to the Riemann problem can be a very time-consuming task because an iterative procedure is needed. Therefore, approximate Riemann solvers are often preferred because they provide satisfactory solutions while using faster algorithms. Two broad families of solvers can be distinguished: (i) solvers where the Riemann problem is simplified (e.g., by linearising the equations), and (ii) solvers where simplified relationships are used to solve the exact problem. The first family of solver includes Roe's solver (Roe, 1981), where the flux at the location of the initial discontinuity is calculated via a wave decomposition under the assumption of a constant Jacobian matrix.

The Jacobian matrix is approximated in such a way that consistency and conservation conditions are satisfied. An entropy fix is needed when a rarefaction wave extends over the location of the initial discontinuity. From another side, primitive variable Riemann solvers (Toro, 1997) use a linearisation of the hyperbolic system with a constant Jacobian matrix in combination with the Rankine–Hugoniot conditions across each wave. This allows a simplified system of equations to be solved for the unknown variables. The Riemann invariants can also be used along characteristics to obtain the simplified system (Lhomme and Guinot, 2007).

12.1.2 Higher Order Accuracy and Higher Dimensions

The cell average of a solution in a cell contains too little information. In order to obtain higher order accuracy, neighbouring cell averages are used to reconstruct an approximate polynomial solution in each cell. This reconstruction procedure is the key step for many high-resolution schemes (Liu et al., 2007). For example, in the Advection-Diffusion-Reaction (ADER) approach (Titarev and Toro, 2002), the numerical flux function is based on the solution of generalised Riemann problems, where the initial data on both sides of the element interfaces are no longer piecewise constant. Here the initial data is a piecewise polynomial, in general separated by a jump at the interface.

The fundamental idea behind the generalised Riemann problem solvers is a temporal Taylor series expansion of the state at the interface, where time derivatives are then replaced by space derivatives using repeatedly the governing conservation