Chapter 5

A_1 del Pezzo surface of degree 6

Let $S \subset \mathbb{P}^6$ be the surface cut out by the intersection of 9 quadrics

$$x_1^2 - x_2 x_4 = x_1 x_5 - x_3 x_4 = x_1 x_3 - x_2 x_5 = x_1 x_6 - x_3 x_5$$

$$= x_2 x_6 - x_3^2 = x_4 x_6 - x_5^2 = x_1 x_4 + x_5 x_7$$

(5.1)

$$= x_1^2 - x_1 x_2 - x_3 x_7 = x_1 x_3 - x_1 x_5 + x_6 x_7 = 0.$$ 

This is the $A_1$ del Pezzo surface of degree 6 that was introduced in (2.21). Any line in $\mathbb{P}^6$ is defined by the intersection of 5 hyperplanes. It is not hard to see that the equations

$$\begin{cases}
    x_1 = x_2 = x_3 = x_5 = x_6 = 0, & x_1 = x_3 = x_4 = x_5 = x_6 = 0, \\
    x_3 = x_5 = x_6 = x_1 - x_4 = x_1 - x_2 = 0,
\end{cases}$$

(5.2)

all define lines contained in $S$. Table 2.6 ensures that these are the only lines contained in $S$. We set $U$ to be the open subset of $S$ obtained by deleting the lines.

Our task in this chapter is to establish the existence of constants $c_1, c_2 \geq 0$ such that

$$N_U(B) = c_1 B \log B^3 + c_2 B \log B^2 + O(B \log B),$$

(5.3)

with

$$c_1 = \frac{\sigma_\infty}{144} \prod_p \left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right)$$

and

$$\sigma_\infty = 6 \int_{\{u,v,t \in \mathbb{R} : 0 < u, ut^2, uv^2, |tv(t-v)| \leq 1\}} dt \, du \, dv.$$ 

This will therefore establish Theorem 2.4.

We begin by translating the counting problem into one in which none of the points counted by $N_U(B)$ are permitted to have zero coordinates.
Lemma 5.1. We have
\[ N_U(B) = 2M(B) + O(B), \]
where \( M(B) \) denotes the number of vectors \( \mathbf{x} \in \mathbb{Z}^7_{\text{prim}} \) such that (5.1) holds, with \( 0 < |x_1|, x_2, x_3, x_4, |x_5|, x_6 \leq B \) and \( |x_7| \leq B \).

Proof. In view of the fact that \( \mathbf{x} \) and \( -\mathbf{x} \) represent the same point in \( \mathbb{P}^6 \), we have
\[ N_U(B) = \frac{1}{2} \# \{ \mathbf{x} \in \mathbb{Z}^7_{\text{prim}} : |\mathbf{x}| \leq B, \ (5.1) \text{ holds but } (5.2) \text{ does not} \}. \]

We need to consider the contribution to the right-hand side from points such that \( x_i = 0 \), for some \( 1 \leq i \leq 7 \). Let us begin by considering the contribution from vectors \( \mathbf{x} \in \mathbb{Z}^7_{\text{prim}} \) for which \( x_1 = 0 \). But then the equations in (5.1) imply that \( x_2x_4 = 0 \). If \( x_2 = 0 \), it is straightforward to check that either \( \mathbf{x} \) satisfies the first system of equations in (5.2), or else \( x_1 = x_2 = x_3 = x_7 = 0 \), \( x_4x_6 = x_5^2 \).

Such points are therefore confined to a plane conic. We therefore obtain \( O(B) \) points overall with \( x_1 = x_2 = 0 \) by Theorem 4.8. If on the other hand, \( x_1 = x_4 = 0 \), then a similar analysis shows that there are \( O(B) \) points in this case too. In view of the first equation in (5.1), the contribution from vectors \( \mathbf{x} \) such that \( x_2x_4 = 0 \) is also \( O(B) \).

Let us now consider the contribution from vectors \( \mathbf{x} \) such that \( x_3 = 0 \) and \( x_1x_2x_4 \neq 0 \). It is easily checked that the only such vectors have \( x_5 = x_6 = 0 \) and \( x_1 - x_4 = x_1 - x_2 = 0 \), and so must lie on a line contained in \( S \). Finally, arguing in a similar fashion, we see that there are no points contained in \( S \) with \( x_5x_6 = 0 \) and \( x_1x_2x_3x_4 \neq 0 \). We have therefore shown that
\[ N_U(B) = \frac{1}{2} \# \{ \mathbf{x} \in \mathbb{Z}^7_{\text{prim}} : x_1 \cdots x_6 \neq 0, \ |\mathbf{x}| \leq B, \ (5.1) \text{ holds} \} + O(B). \]

We would now like to restrict our attention to positive values of \( x_2, x_3, x_4, x_6 \). The equations for \( S \) imply that \( x_2, x_4, x_6 \) all share the same sign. On absorbing the minus sign into \( x_1 \) there is a clear bijection between solutions to (5.1) with \( x_2, x_4, x_6 < 0 \) and solutions with \( x_2, x_4, x_6 > 0 \). We choose to count the latter. Arguing similarly, by absorbing the minus signs into \( x_5 \) and \( x_7 \), we see that there is a bijection between the solutions to (5.1) with \( x_3 < 0 \) and \( x_2, x_4, x_6 > 0 \), and the solutions with \( x_2, x_3, x_4, x_6 > 0 \). Fixing our attention on the latter set of points, we therefore complete the proof of Lemma 5.1. \( \square \)

### 5.1 Passage to the universal torsor

Let \( \tilde{S} \) denote the minimal desingularisation of the surface \( S \). By determining the Cox ring, Derenthal [53] has calculated the universal torsor above \( \tilde{S} \). In this setting it is defined by a single equation
\[ s_1y_1 - s_2y_2 + s_3y_3 = 0, \quad (5.4) \]