Ergodic Theorems for Homogeneous Dilations

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Abstract. In this paper we prove a general ergodic theorem for ergodic and measure-preserving actions of $\mathbb{R}^d$ on standard Borel spaces. In particular, we cover R.L. Jones’ ergodic theorem on spheres. Our main theorem is concerned with almost everywhere convergence of ergodic averages with respect to homogeneous dilations of certain Rajchman measures on $\mathbb{R}^d$. Applications include averages over smooth submanifolds and polynomial curves.

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1. Introduction

The first multidimensional pointwise ergodic theorem is due to N. Wiener [19], who proved that if $(X, \mathcal{B}, \mu)$ is a standard Borel space and $T$ is an ergodic measure-preserving action of $\mathbb{R}^d$ on $X$, then for all $f \in L^1(X)$, the limit

$$\lim_{\lambda \to \infty} \frac{1}{|B|} \int_B f(T_{\lambda t}x) \, dt = \int_X f \, d\mu,$$

exists almost everywhere on $X$, where $B$ denotes the unit ball in $\mathbb{R}^d$ and $|B|$ the volume of $B$. It is not hard to extend this theorem to the more general setting where the normalized characteristic function of a ball is replaced by an absolutely continuous probability measure on $\mathbb{R}^d$.

In this paper we prove Wiener’s ergodic theorem with respect to not necessarily absolutely continuous probability measures on $\mathbb{R}^d$; more precisely, we are interested in the class $\mathcal{C}_p$ of probability measures $\nu$ for which

$$\lim_{\lambda \to \infty} \int_{\mathbb{R}^d} f(T_{\lambda t}x) \, d\nu(t) = \int_X f \, d\mu$$

exists almost everywhere on $X$ for all $f$ in $L^p(X)$, where the range of $p$ is allowed to depend on $\nu$. It is obviously necessary that $\nu$ is continuous, i.e., does not give positive mass to individual points, for this to true. We will say that Wiener’s
ergodic theorem holds for $\nu$ if the limit above exists almost everywhere on $X$. It was proved by R.L. Jones [9] that the induced Lebesgue measure on $S^{d-1}$ in $\mathbb{R}^d$ for $d \geq 3$ belongs to the class $C_p$ for $p > \frac{d}{d-1}$. This was later extended to $d = 2$ by M. Lacey [11]. We will extend their result to a much larger class of measures. We stress however that the techniques in this paper are not new, and many of the results were probably already known to the experts. We hope that the paper will serve as a survey of some classical ideas in the harmonic analysis approach to ergodic theory.

Recall that the Fourier dimension of a probability measure $\nu$ on $\mathbb{R}^d$ is defined as the supremum over all $0 \leq a \leq d$ such that

$$|\hat{\nu}(\xi)| \leq C|\xi|^{-a/2} \quad \text{as} \quad \xi \to \infty.$$  

For instance, if $S$ is a smooth hypersurface in $\mathbb{R}^d$ with non-vanishing Gaussian curvature and $\nu$ is the induced Lebesgue measure on $S$, then the Fourier dimension of $\nu$ is at least $n - 1$ [15], which motivates the terminology. Note that there are many non-smooth sets (e.g., random Cantor sets [2]) in $\mathbb{R}^d$ which support probability measures with high Fourier dimension. However, by Frostman’s lemma (see, e.g., [12]), the Fourier dimension is always bounded from above by the Hausdorff dimension of the support of $\nu$.

In this paper we prove Wiener’s ergodic theorem for probability measures $\nu$ on $\mathbb{R}^d$ with sufficiently large Fourier dimensions.

**Theorem 1.1.** Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability measure space, and suppose $T$ is an ergodic Borel measurable action of $\mathbb{R}^d$ on $X$ which preserves $\mu$. If $\nu$ is a compactly supported probability measure on $\mathbb{R}^d$ with Fourier dimension $a > 1$, then

$$\lim_{\lambda \to \infty} \int_{\mathbb{R}^d} f(T_{\lambda t}x) \, d\nu(t) = \int_X f \, d\mu$$

almost everywhere on $X$, for all $f$ in $L^p(X)$ for $p > p_a$, where

$$p_a = \frac{1 + a}{a}.$$ 

The question of mean convergence is much simpler and an immediate consequence of the spectral theorem for unitary operators. Recall that a probability measure $\nu$ on $\mathbb{R}^d$ is a Rajchman measure if the Fourier transform of $\nu$ decays to zero at infinity. Note that a Rajchman measure is always continuous, but not necessarily absolutely continuous. Indeed, there are Rajchman measures supported on the set of Liouville numbers in $\mathbb{R}$ [3], which have zero Hausdorff dimension. The following proposition is well known and we only include it for completeness.

**Proposition 1.2.** Let $U$ be a unitary representation of $\mathbb{R}^d$ on a separable Hilbert space $H$. Let $\rho$ be a Rajchman measure on $\mathbb{R}^d$, and define for $x \in H$, the operator

$$A_\lambda x = \int_{\mathbb{R}^d} U_{\lambda t} x \, d\rho(t), \quad \lambda > 0.$$