Chapter V
Spectra of fractal pseudodifferential operators

26 Introduction and preliminaries

26.1 The Weyl problem

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n$ and let $-\Delta$ be the Dirichlet Laplacian considered in the Hilbert space $L^2(\Omega)$ with its domain of definition

$$\text{dom}(-\Delta) = H^2(\Omega) \cap \overset{\circ}{H}^1(\Omega)$$

according to (25.28). Recall that $H^2(\Omega)$ is the restriction of $H^2(\mathbb{R}^n) = B_{2,2}^2(\mathbb{R}^n)$ on $\Omega$ as introduced in (23.1) and that $\overset{\circ}{H}^1(\Omega)$ is the completion of $C_0^\infty(\Omega) = D(\Omega)$ in the norm given by (25.20). Let

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \to \infty \quad \text{if} \quad j \to \infty,$$

be the ordered eigenvalues with respect to their multiplicities. The counting function $N(\lambda)$ is given by

$$N(\lambda) = \#\{\lambda_j < \lambda\}, \quad \lambda > 0,$$

the number of eigenvalues less than $\lambda$. Under some additional geometric assumptions one has

$$N(\lambda) = (2\pi)^{-n} \omega_n \text{vol}(\Omega) \lambda^n - \kappa_n \text{vol}(\partial \Omega) \lambda^{n-1} (1 + o(1))$$

as $\lambda \to \infty$, where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$, and $\kappa_n$ is some positive number depending only on $n$. The equivalence (25.27) with $\lambda_k$ in place $\mu_k$ is an immediate consequence of (26.4). Historical remarks and references may be found in the papers mentioned at the beginning of 25.6. We do not go into detail. It is well-known that this problem goes back to H. Weyl in 1911/12 and has attracted much attention up to our time. If $n = 2$ then $\sqrt{\lambda_j}$ can be interpreted as the eigenfrequencies of a drum with the vibrating membrane $\Omega$.

26.2 Fractal drums

One may ask what happens with the counting function $N(\lambda)$ or its weaker version
\[ \lambda_k \sim k^{\frac{2}{n}}, \quad k \in \mathbb{N}, \quad (26.5) \]
if the smooth bounded domain $\Omega$ in $\mathbb{R}^n$ is replaced by something bounded, but non-smooth. If $n = 2$ or also $n = 3$ this may be related as above to the eigenfrequencies of a suitable membrane. There are two versions in literature and we add a third one.

(i) Let $\Omega$ be again a bounded domain in $\mathbb{R}^n$ and let $\partial \Omega$ be a fractal with the Hausdorff dimension $d$, where $n - 1 < d < n$. Then $-\Delta$ makes sense at least according to 25.3 and 25.4. M. V. Berry conjectured that one has again (26.4) with the same main term on the right-hand side and $\mathcal{H}^d(\partial \Omega) \lambda^{\frac{d}{2}}$ in place of $\text{vol}(\partial \Omega) \lambda^{\frac{n-1}{2}}$. This bold conjecture cannot be correct in this generality. However if $\partial \Omega$ is a compact $d$-set according to Definition 3.1 then a partial affirmative answer was given in [Lap91], where one finds a proof of (26.4) with unchanged main term and the remainder term $O(\lambda^{\frac{d}{2}})$ on the right-hand side. We refer to this paper and also to [Lap93], [LeV96], [Vas91], [FlV93], and [FLV95] for a thorough discussion of these problems.

(ii) Even bolder is the assumption that both $\Omega$ and $\partial \Omega$ are fractals, describing, say, the vibrating earth with its fractal core and its fractal surface (whatever this means). But then one has to say what is meant by the Laplacian on $\Omega$. This is just the problem we addressed in 25.6 and the references given there reflect the state of art.

(iii) Our intention here is different from that one in (i) or in (ii). Let $(-\Delta)^{-1}$ be the inverse of the Dirichlet Laplacian $-\Delta$ in a smooth bounded domain $\Omega$ in $\mathbb{R}^n$ considered in (i). For its eigenvalues $\mu_k = \lambda_k^{-1}$ we have
\[ \mu_k \sim k^{-\frac{2}{n}}, \quad k \in \mathbb{N}. \quad (26.6) \]
Recall that $\sim$ means that there are two constants $0 < c_1 \leq c_2 < \infty$ such that
\[ c_1 k^{-\frac{2}{n}} \leq \mu_k \leq c_2 k^{-\frac{2}{n}}, \quad k \in \mathbb{N}. \quad (26.7) \]
If $n = 2$ then $(-\Delta)^{-1}$ represents in the above-described way a vibrating membrane where the mass is evenly distributed, that means the mass density $m(x)$, $x \in \Omega$, is constant. If this is not so, then it follows by the usual physical reasoning that $(-\Delta)^{-1}$ must be replaced by
\[ B : f \mapsto (-\Delta)^{-1} m(x) f. \quad (26.8) \]
Assuming that the whole mass is concentrated on a compact fractal $\Gamma$ with $\Gamma \subset \Omega$. Say, $\Gamma$ is a compact $d$-set according to Definition 3.1 with the measure $\mu$. Then $B$ is given by, say,
\[ B = (-\Delta)^{-1} b(\gamma) \mu, \quad b(\gamma) \in L_p(\Gamma), \quad (26.9) \]