Lecture 6

Examples of derived algebraic stacks

In this last lecture, we present examples of derived algebraic stacks.

6.1 The derived moduli space of local systems

We come back to the example that we presented in the first lecture, namely the moduli problem of linear representations of a discrete group. We will now reconsider it from the point of view of derived algebraic geometry. We will try to treat this example in some detail, as we think it is a rather simple, but interesting, example of a derived algebraic stack.

A linear representation of a group $G$ can also be interpreted as a local system on the space $BG$. We will therefore study the moduli problem from this topological point of view. We fix a finite CW-complex $X$ and we are going to define a derived stack $\mathbb{R}\text{Loc}(X)$ classifying local systems on $X$. We will see that this stack is an algebraic derived 1-stack and we will describe its higher tangent spaces in terms of cohomology groups of $X$. When $X = BG$ for a discrete group $G$, the derived algebraic stack $\mathbb{R}\text{Loc}(X)$ is the correct moduli space of linear representations of $G$.

We start by considering the non-derived algebraic 1-stack $\text{Vect}$ classifying projective modules of finite type. By definition, $\text{Vect}$ sends a commutative ring $A$ to the nerve of the groupoid of projective $A$-modules of finite type. The stack $\text{Vect}$ is a 1-stack. It is easy to see that $\text{Vect}$ is an algebraic 1-stack. Indeed, we have a decomposition

$$\text{Vect} \simeq \coprod_n \text{Vect}_n,$$

where $\text{Vect}_n \subset \text{Vect}$ is the substack of projective modules of rank $n$ (recall that a projective $A$-module of finite type $M$ is of rank $n$ if, for any field $K$ and any
morphism $A \to K$, the $K$-vector space $M \otimes_A K$ is of dimension $n$). It is therefore enough to prove that $\text{Vect}_n$ is an algebraic 1-stack. This last statement will itself follow from the identification

$$\text{Vect}_n \simeq [*/\text{Gl}_n] = B\text{Gl}_n,$$

where $\text{Gl}_n$ is the affine group scheme sending $A$ to $\text{Gl}_n(A)$. In order to prove that $\text{Vect}_n \simeq B\text{Gl}_n$, we construct a morphism of simplicial presheaves

$$B\text{Gl}_n \to \text{Vect}_n$$

by sending the base point of $B\text{Gl}_n$ to the trivial projective module of rank $n$. For a given commutative ring $A$, the morphism $B\text{Gl}_n(A) \to \text{Vect}_n(A)$ sends the base point to $A^n$ and identifies $\text{Gl}_n(A)$ with the automorphism group of $A^n$. The claim is that this morphism induces isomorphisms on all higher homotopy sheaves, it only remains to show that it induces an isomorphism on the sheaves $\pi_0$. But this in turn follows from the fact that $\pi_0(\text{Vect}_n) \simeq *$, because any projective $A$-module of finite type is locally free for the Zariski topology on $\text{Spec} A$.

The algebraic stack $\text{Vect}$ is now considered as an algebraic derived stack using the inclusion functor $j : \text{Ho}(s\text{Pr}(\text{Aff})) \to \text{Ho}(\text{dAff}^\sim)$. We consider a fibrant model $F \in \text{dAff}^\sim$ for $j(\text{Vect})$, and we define a new simplicial presheaf

$$\mathbb{R}\text{Loc}(X) : \text{dAff}^{\text{op}} \to \text{sSet}$$

which sends $A \in s\text{Comm}$ to $\text{Map}(X, |F(A)|)$, the simplicial set of continuous maps from $X$ to $|F(A)|$.

**Definition 6.1.1.** The derived stack $\mathbb{R}\text{Loc}(X)$ defined above is called the *derived moduli stack of local systems* on $X$.

We will now describe some basic properties of the derived stack $\mathbb{R}\text{Loc}(X)$. We start by a description of its classical part $h^0(\mathbb{R}\text{Loc}(X))$, which will show that it does classify local systems on $X$. We will then show that $\mathbb{R}\text{Loc}(X)$ is an algebraic derived stack locally of finite presentation over $\text{Spec Z}$, and that it can be written as

$$\mathbb{R}\text{Loc}(X) \simeq \coprod_n \mathbb{R}\text{Loc}_n(X)$$

where $\mathbb{R}\text{Loc}_n(X)$ is the part classifying local systems of rank $n$ and is itself strongly of finite type. Finally, we will compute its tangent spaces in terms of the cohomology of $X$.

For $A \in \text{Comm}$, note that $h^0(\mathbb{R}\text{Loc}(X))(A)$ is by definition the simplicial set $\text{Map}(X, |F(A)|)$. Now, $F(A)$ is a fibrant model for $j(\text{Vect})(A) \simeq \text{Vect}(A)$, and