Chapter 15

A Brief Survey of Integration

In this chapter we briefly review some notions and results related to integration. We in particular discuss the following topics:

1. Algebras and $\sigma$-algebras.
2. Measurable functions.
4. The main convergence theorems.
5. Complete measures.

Although we use in this book integration theory only on the real line, we have chosen to present the essentials of the general theory. We recommend to the interested student the books of Rudin [124] and of Folland [57] for a complete, but relatively short, discussion of integration. The industrious might want to look at the series of books of Bourbaki on integration.

15.1 Introduction

Students who begin to learn complex analysis usually have only a knowledge of the Riemann integral, and not of the general theory of integration. This is certainly good enough to define large families of functions via integrals (see for instance Exercises 4.4.14 and 4.4.15). But Riemann integrable functions do not have nice properties with respect to limits, as is illustrated in the following exercise.

Exercise 15.1.1. Give an example of a uniformly bounded sequence of functions $(f_n)_{n \in \mathbb{N}}$ which are Riemann integrable in the interval $[0,1]$, which converge pointwise, but whose pointwise limit is not Riemann integrable.

It is therefore difficult to have a general theorem which allows interchanging limit and integration for Riemann integrable functions. One such theorem is
Weierstrass’ theorem (see Theorem 12.4.1), but the hypothesis of uniform convergence in that theorem is very strong. Furthermore, Weierstrass’s theorem concerns only continuous functions with compact support. A number of interesting examples, such as the Gamma function, are defined in terms of integrals of continuous functions on an infinite interval.

The above discussion gives a first motivation to go beyond the Riemann integral. Another, and related, motivation is as follows. The space of, say continuous, complex-valued functions defined on a compact interval \([a, b]\), and endowed with the metric

\[
d(f, g) = \left( \int_a^b |f(t) - g(t)|^2 dt \right)^{1/2}
\]

is not complete. See Exercise 14.1.4, and the discussion after the proof of this exercise. From the general theory of metric spaces we know that it is isometrically included in a complete metric space, unique up to an isometry of metric spaces. For the problems at hand in signal processing and in the theory of linear systems, this abstract completion is not too useful. It is more appropriate to consider the Lebesgue spaces \(L_2(a, b)\). The construction of these spaces is one of the keystones of the integration theory which we review briefly in this chapter. For engineers, the Lebesgue spaces \(L_2(\mathbb{R}, dx)\) play a fundamental role as spaces of signals with finite energy. These are of course not the only motivation and advantages of modern integration theory. A third, and very important, motivation to introduce measure theory in the study of analytic functions, is the study of boundary behaviour of a function. For instance, and this is quite beyond the scope of the present book, a function (say \(f\)) which is analytic and bounded in the open unit disk admits almost everywhere radial (and in fact non-tangential) boundary values. For radial limit, this means that, at the possible exception of a subset of \([0, 2\pi]\) of Lebesgue measure zero, the limit

\[
\lim_{r \uparrow 1} f(re^{i\theta})
\]

exists.

Another very important fact, not touched upon here, is Riesz’s theorem on the dual of the space of continuous functions on a locally compact Hausdorff space. The space \(C_0[0, 1]\), endowed with the maximum norm

\[
\|x\|_\infty = \max_{t \in [0, 1]} |x(t)|
\]

is a Banach space, and it is a natural question to ask what is its dual, that is, to describe the set of its linear continuous functionals. For instance

\[
\varphi(x) = x(0),
\]

and

\[
\varphi(x) = \int_0^1 x(t)x(t)\varphi(t)dt
\]