Chapter 1

The Laplace Integral

The first three sections of this chapter are of a preliminary nature. There, we collect properties of the Bochner integral of functions of a real variable with values in a Banach space $X$. We then concentrate on the basic properties of the Laplace integral

$$\hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t) \, dt := \lim_{\tau \to \infty} \int_0^\tau e^{-\lambda t} f(t) \, dt$$

for locally Bochner integrable functions $f : \mathbb{R}_+ \to X$. In Section 1.4 we describe the set of complex numbers $\lambda$ for which the Laplace integral converges. It will be shown that the domain of convergence is non-empty if and only if the antiderivative of $f$ is of exponential growth. In Section 1.5 we discuss the holomorphy of $\lambda \mapsto \hat{f}(\lambda)$ and in Section 1.7 we show that $f$ is uniquely determined by the Laplace integrals $\hat{f}(\lambda)$ (uniqueness and inversion). In Section 1.6 we prove the operational properties of the Laplace integral which are essential in applications to differential and integral equations. In particular, we show that the Laplace integral of the convolution $k \ast f : t \mapsto \int_0^t k(t-s)f(s) \, ds$ of a scalar-valued function $k$ with a vector-valued function $f$ is given by

$$(k \ast \hat{f})(\lambda) = \hat{\check{k}}(\lambda) \hat{f}(\lambda)$$

if $\hat{f}(\lambda)$ exists and $\hat{\check{k}}(\lambda)$ exists as an absolutely convergent integral. In Section 1.8 we consider vector-valued Fourier transforms and we show that Plancherel’s theorem and the Paley-Wiener theorem extend to functions with values in a Hilbert space. Finally, after introducing the basic properties of the Riemann-Stieltjes integral in Section 1.9, we extend in Section 1.10 the basic properties of Laplace integrals to Laplace-Stieltjes integrals

$$\hat{\mathcal{d}F}(\lambda) := \int_0^\infty e^{-\lambda t} \, dF(t) := \lim_{\tau \to \infty} \int_0^\tau e^{-\lambda t} \, dF(t)$$

of functions $F$ of bounded semivariation.
If $f$ is Bochner integrable, then the normalized antiderivative $t \mapsto F(t) := \int_0^t f(s) \, ds$ is of bounded variation. We will see that $\hat{f}(\lambda)$ exists if and only if $\hat{dF}(\lambda)$ exists, and in this case $\hat{f}(\lambda) = \hat{dF}(\lambda)$. Thus, the Laplace-Stieltjes integral is a natural extension of the Laplace integral. This extension is crucial for our discussion of the Laplace transform in Chapter 2 since there are many functions $r : (\omega, \infty) \to X$ which can be represented as a Laplace-Stieltjes integral, but not as a Laplace integral of a Bochner integrable function. Examples are, among others, Dirichlet series $r(\lambda) = \sum_{n=1}^{\infty} a_n e^{-\lambda n} = \hat{dF}(\lambda)$, where $F$ is the step function $\sum_{n=1}^{\infty} a_n \chi_{(n, \infty)}$, or any function $r(\lambda) = \hat{dF}(\lambda)$, where $F$ is of bounded semivariation, but not the antiderivative of a Bochner integrable function.

1.1 The Bochner Integral

This section contains some properties of the Bochner integral of vector-valued functions. We shall consider only those properties which are used in later sections, and we shall assume that the reader is familiar with the basic facts about measure and integration of scalar-valued functions.

Let $X$ be a complex Banach space, and let $I$ be an interval (bounded or unbounded) in $\mathbb{R}$, or a rectangle in $\mathbb{R}^2$. A function $f : I \to X$ is simple if it is of the form $f(t) = \sum_{r=1}^{n} x_r \chi_{\Omega_r}(t)$ for some $n \in \mathbb{N} := \{1, 2, \ldots\}, x_r \in X$ and Lebesgue measurable sets $\Omega_r \subset I$ with finite Lebesgue measure $m(\Omega_r)$; $f$ is a step function when each $\Omega_r$ can be chosen to be an interval, or a rectangle in $\mathbb{R}^2$. Here $\chi_{\Omega}$ denotes the characteristic (indicator) function of $\Omega$. In the representation of a simple function, the sets $\Omega_r$ may always be arranged to be disjoint, and then

$$f(t) = \begin{cases} x_r & (t \in \Omega_r; r = 1, 2, \ldots, n) \\
0 & \text{otherwise.} \end{cases}$$

A function $f : I \to X$ is measurable if there is a sequence of simple functions $g_n$ such that $f(t) = \lim_{n \to \infty} g_n(t)$ for almost all $t \in I$. Since any $\chi_{\Omega}$ for $\Omega$ measurable is a pointwise almost everywhere (a.e.) limit of a sequence of step functions, it is not difficult to see that the functions $g_n$ can be chosen to be step functions. When $X = \mathbb{C}$, this definition agrees with the usual definition of (Lebesgue) measurable functions. It is easy to see that if $f : I \to X$, $g : I \to X$ and $h : I \to \mathbb{C}$ are measurable, then $f + g$ and $h \cdot f$ are measurable. Moreover, if $k : X \to Y$ is continuous (where $Y$ is any Banach space), then $k \circ f$ is measurable whenever $f$ is measurable. In particular, $\|f\|$ is measurable. If $X$ is a closed subspace of $Y$, and $f$ is measurable as a $Y$-valued function, then $f$ is also measurable as an $X$-valued function.

To verify measurability of a function we often use the characterization given by Pettis's theorem below. We say that $f : I \to X$ is countably valued if there is a countable partition $\{\Omega_n : n \in \mathbb{N}\}$ of $I$ into subsets $\Omega_n$ such that $f$ is constant on each $\Omega_n$; it is easy to see that $f$ is measurable if each $\Omega_n$ is measurable (and