Chapter 2

The Laplace Transform

In this chapter the emphasis of the discussion shifts from Laplace integrals \( \hat{f}(\lambda) \) and \( d\hat{F}(\lambda) \) to the Laplace transform \( \mathcal{L} : f \mapsto \hat{f} \) and to the Laplace-Stieltjes transform \( \mathcal{L}_S : F \mapsto dF \). The Laplace transform is considered first as an operator acting on \( L^\infty(\mathbb{R}_+, X) \) and the Laplace-Stieltjes transform as an operator on

\[
\text{Lip}_0(\mathbb{R}_+, X) := \left\{ F : \mathbb{R}_+ \to X : F(0) = 0, \|F\|_{\text{Lip}_0(\mathbb{R}_+, X)} := \sup_{t,s \geq 0} \frac{\|F(t) - F(s)\|}{|t - s|} < \infty \right\}.
\]

These domains of \( \mathcal{L} \) and \( \mathcal{L}_S \) are relatively easy to deal with and have immediate and important applications to abstract differential and integral equations.

The following observation is the key to one of the basic structures of Laplace transform theory. If \( f \in L^\infty(\mathbb{R}_+, X) \), then \( t \mapsto F(t) := \int_0^t f(s) \, ds \) belongs to \( \text{Lip}_0(\mathbb{R}_+, X) \) and

\[
\mathcal{L}(f)(\lambda) = \int_0^\infty e^{-\lambda t} f(t) \, dt = \int_0^\infty e^{-\lambda t} dF(t) = T_F(e^{-\lambda}),
\]

where \( T_F : g \mapsto \int_0^\infty g(s) \, dF(s) \) is a bounded linear operator from \( L^1(\mathbb{R}_+) \) into \( X \), and where \( e^{-\lambda} \) denotes the exponential function \( t \mapsto e^{-\lambda t} \). The operator \( T_F \) is fundamental to Laplace transform theory. In Section 2.1 it is shown that \( \Phi_S : F \mapsto T_F \) is an isometric isomorphism between \( \text{Lip}_0(\mathbb{R}_+, X) \) and \( \mathcal{L}(L^1(\mathbb{R}_+), X) \) (Riesz-Stieltjes representation theorem). This representation is crucial for the following reason. The main purpose of Laplace transform theory is to translate properties of the generating function \( F \) into properties of the resulting function \( \lambda \mapsto r(\lambda) = \int_0^\infty e^{-\lambda t} dF(t) \) and vice versa. Since \( F(t) = T_F \chi_{[0,t]} = \int_0^\infty \chi_{[0,t]}(s) \, dF(s) \) and \( r(\lambda) = T_F e^{-\lambda} = \int_0^\infty e^{-\lambda s} \, dF(s) \), the generating function \( F \) as well as the resulting function \( r \) are evaluations of the same bounded linear operator acting on different total subsets of \( L^1(\mathbb{R}_+) \).

In Section 2.2, the range of the Laplace-Stieltjes transform acting on Lip_0(\(\mathbb{R}_+\), \(X\)) is characterized. It is shown that a function \(r : \mathbb{R}_+ \rightarrow X\) has a Laplace-Stieltjes representation \(r = \mathcal{L}_S(F)\) for some \(F \in \text{Lip}_0(\mathbb{R}_+, X)\) if and only if \(r\) is a \(C^\infty\)-function whose Taylor coefficients satisfy the estimate

\[
\|r\|_W := \sup_{n \in \mathbb{N}_0} \sup_{\lambda > 0} \frac{\lambda^{n+1}}{n!} \|r^{(n)}(\lambda)\| < \infty. \tag{2.1}
\]

This can be rephrased by saying that the Laplace-Stieltjes transform is an isometric isomorphism between the Banach spaces Lip_0(\(\mathbb{R}_+, X\)) and

\[
C^\infty_W((0, \infty), X) := \{r \in C^\infty((0, \infty), X) : \|r\|_W < \infty\}.
\]

If the Banach space \(X\) has the Radon-Nikodym property (see Section 1.2), then (and only then) “Widder’s growth conditions” (2.1) are necessary and sufficient for \(r\) to have a Laplace representation \(r = \mathcal{L}(f)\) for some \(f \in L^\infty(\mathbb{R}_+, X)\); i.e., Banach spaces with the Radon-Nikodym property are precisely those Banach spaces in which the Laplace transform is an isometric isomorphism between \(L^\infty(\mathbb{R}_+, X)\) and \(C^\infty_W((0, \infty), X)\). For \(X = \mathbb{C}\), this is a classical result usually known as “Widder’s Theorem”.

If \(r = \mathcal{L}_S(F)\) for some \(F \in \text{Lip}_0(\mathbb{R}_+, X)\), then the inverse Laplace-Stieltjes transform has many different representations. A few of them, such as

\[
F(t) = \frac{1}{2\pi i} \int e^{\lambda t} \frac{r(\lambda)}{\lambda} \, d\lambda = \lim_{n \to \infty} \sum_{j=1}^{\infty} (-1)^{j+1} e^{tnj} r(nj)
\]

will be proved in Section 2.3.

In Section 2.4, the results of the previous sections are extended to functions with exponential growth at infinity; i.e., we investigate the Laplace transform acting on functions \(f\) with \(\text{ess sup}_{t \geq 0} \|e^{-\omega t} f(t)\| < \infty\).

In applications it is usually impossible to verify whether or not a given function \(r\) satisfies Widder’s growth conditions (2.1). Thus, in Sections 2.5 and 2.6 some complex growth conditions are discussed which are necessary (and in a certain sense sufficient) for a holomorphic function \(r : \{\Re \lambda > \omega\} \rightarrow X\) to have a Laplace representation. In Section 2.5, the growth condition considered is

\[
\sup_{\Re \lambda > \omega} \|\lambda^{1+b} r(\lambda)\| < \infty
\]

for some \(b > 0\).

In Section 2.6, we discuss functions \(r\) which are holomorphic in a sector \(\Sigma := \{\arg(\lambda) < \frac{\pi}{2} + \varepsilon\}\) and satisfy \(\sup_{\lambda \in \Sigma} \|\lambda r(\lambda)\| < \infty\). We will see that any such \(r\) is the Laplace transform of a function which is holomorphic in the sector \(\{\arg(\lambda) < \varepsilon\}\). The final class of functions which we will consider are the completely monotonic ones; i.e., \(C^\infty\)-functions \(r\) with values in an ordered Banach space such that \((-1)^n r^{(n)}(\lambda) \geq 0\) for all \(n \in \mathbb{N}_0\) and \(\lambda > \omega\). In the scalar case,