Chapter 4

Asymptotics of Laplace Transforms

Frequently, convergence of a function $f : \mathbb{R}_+ \to X$ for $t \to \infty$ implies convergence of an average of this function. Assertions of this type are called *Abelian theorems*. A theorem is called *Tauberian* if, conversely, convergence of the function is deduced from the convergence of an average.

The Abelian theorems which we present in Section 4.1 are quite easy to prove. However, the Tauberian theorems corresponding to their converse versions are much more delicate. They need additional hypotheses, so-called *Tauberian conditions*. Section 4.2 is devoted to Tauberian conditions of real type (for example, boundedness or positivity of $f$).

Interesting applications of these Abelian and real Tauberian theorems to semigroups are given in Section 4.3 where mean ergodicity is discussed. This interrupts the general theme of this chapter, but the results will be useful in the subsequent sections where the notion of mean ergodicity is needed.

In Section 4.4 a complex Tauberian theorem is proved with the help of an elegant contour argument. Here we make assumptions on holomorphic extensions of $f$ on the imaginary axis. We restrict ourselves to the case of one singularity in order to keep the ideas more transparent, but this case is already of special interest. For example, an immediate consequence is Gelfand’s theorem, saying that a bounded $C_0$-group is trivial (i.e., the identity) if and only if the spectrum of its generator is reduced to $\{0\}$.

One interesting type of asymptotic behaviour for large time is almost periodicity. The concept is introduced in Section 4.5 where elementary properties are proved for functions on $\mathbb{R}$. In Section 4.6, Loomis’s theorem and its vector-valued version are proved by an elegant quotient method which allows one to apply Gelfand’s theorem. The basic notion is the Carleman spectrum for a bounded measurable function defined on the line, and Loomis’s theorem states that any
bounded uniformly continuous function $f : \mathbb{R} \to \mathbb{C}$ with countable spectrum is almost periodic. There is one vector-valued version of Loomis’ theorem, which is valid for every Banach space and involves an ergodicity condition. The other vector-valued version holds without further assumptions on the function but a geometric condition on the Banach space is needed.

Functions on the half-line are considered in Section 4.7. The naturally associated “half-line” spectrum is discussed and the main theorem is a complex Tauberian theorem for functions with countable spectrum, which is proved by the same technique as we proved Loomis’s theorem.

In Section 4.8 we come back to functions defined on the line showing that the Carleman spectrum (defined by holomorphy) and the Beurling spectrum (defined by the Fourier transform) coincide. This allows us to prove a very general complex Tauberian theorem for functions on the half-line in Section 4.9. Here we use Fourier transform methods which allow one to reduce the problem to an application of Loomis’s theorem (in the scalar case).

The structure of this chapter needs some explanation in view of our main purpose, namely the proof of a complex Tauberian theorem on the half-line. We present three different methods by which we prove the result in increasing generality (Theorems 4.4.8, 4.7.7 and 4.9.7); namely, the contour method, the quotient method and the Fourier method. If Loomis’s theorem (Corollary 4.6.4) is accepted, the Fourier method of Section 4.9 is the most general. The contour method, presented as the first approach in Section 4.4, is the most elementary. It gives us Gelfand’s theorem (Corollary 4.4.12) and other interesting consequences. The quotient method uses Gelfand’s theorem. It gives us Loomis’s theorem on the line (Section 4.6), and in an elegant way the fairly general complex Tauberian Theorem 4.7.7 on the half-line.

### 4.1 Abelian Theorems

Throughout this section, $f$ denotes a function in $L^1_{\text{loc}}(\mathbb{R}_+, X)$, where $X$ is a Banach space and $\mathbb{R}_+ := [0, \infty)$ is the right half-line.

We consider the following three types of averages.

**Definition 4.1.1.** Let $f_\infty \in X$. We say that

a) $f(t)$ converges to $f_\infty$ in the sense of Abel as $t \to \infty$ if $\text{abs}(f) \leq 0$ and

$$A\text{-}\lim_{t \to \infty} f(t) := \lim_{\lambda \downarrow 0} \lambda \hat{f}(\lambda) = f_\infty;$$

b) $f(t)$ is B-convergent to $f_\infty$ as $t \to \infty$, or simply write $B\text{-}\lim_{t \to \infty} f(t) = f_\infty$, if for every $\delta > 0$, $\lim_{t \to \infty} \frac{1}{\delta} \int_t^{t+\delta} f(s) \, ds = f_\infty$;

c) $f(t)$ converges to $f_\infty$ in the sense of Cesàro as $t \to \infty$ if

$$C\text{-}\lim_{t \to \infty} f(t) := \lim_{t \to \infty} \frac{1}{t} \int_0^t f(s) \, ds = f_\infty.$$