Chapter 1

The Finite Fourier Transform

A good starting point is the finite Fourier transform that underpins the contents of the first thirteen chapters of the book.

Let \( \mathbb{C} \) be the set of all complex numbers. For a positive integer \( N \geq 2 \), we let \( \mathbb{C}^N \) be the set defined by

\[
\mathbb{C}^N = \left\{ \left( \begin{array}{c} z(0) \\ z(1) \\ \vdots \\ z(N-1) \end{array} \right) : z(n) \in \mathbb{C}, n = 0, 1, \ldots, N-1 \right\}.
\]

Then \( \mathbb{C}^N \) is an \( N \)-dimensional complex vector space with respect to the usual addition and scalar multiplication of vectors. In fact, it is an inner product space in which the inner product \( (\ , \ ) \) and norm \( \| \| \) are defined by

\[
(z, w) = \sum_{n=0}^{N-1} z(n)\overline{w(n)}
\]

and

\[
\|z\|^2 = (z, z) = \sum_{n=0}^{N-1} |z(n)|^2
\]

for all \( z = \left( \begin{array}{c} z(0) \\ z(1) \\ \vdots \\ z(N-1) \end{array} \right) \) and \( w = \left( \begin{array}{c} w(0) \\ w(1) \\ \vdots \\ w(N-1) \end{array} \right) \) in \( \mathbb{C}^N \). Of particular importance in the first thirteen chapters is the space \( \mathbb{Z}^N \) defined by

\[
\mathbb{Z}^N = \{0, 1, \ldots, N-1 \}.
\]
Let $z : \mathbb{Z}_N \to \mathbb{C}$ be a function. Then the function $z : \mathbb{Z}_N \to \mathbb{C}$ is completely specified by
\[
\begin{pmatrix}
z(0) \\
z(1) \\
\vdots \\
z(N-1)
\end{pmatrix}.
\]
Thus, we can write
\[
z = \begin{pmatrix}
z(0) \\
z(1) \\
\vdots \\
z(N-1)
\end{pmatrix}.
\]
In other words, we can think of the function $z : \mathbb{Z}_N \to \mathbb{C}$ as a finite sequence. If we let $L^2(\mathbb{Z}_N)$ be the set of all finite sequences, then we get
\[
L^2(\mathbb{Z}_N) = \mathbb{C}^N.
\]
Thus, $\mathbb{C}^N$ can be considered as the set of all finite sequences, or more precisely, functions on $\mathbb{Z}_N$. These finite sequences, i.e., functions on $\mathbb{Z}_N$, are in fact the mathematical analogs of digital signals in electrical engineering.

**Definition 1.1.** Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_{N-1} \in L^2(\mathbb{Z}_N)$ be defined by
\[
\epsilon_m = \begin{pmatrix}
0 \\
0 \\
\vdots \\
1
\end{pmatrix}, \quad m = 0, 1, \ldots, N-1,
\]
where $\epsilon_m$ has 1 in the $m^{th}$ position and zeros elsewhere.

**Proposition 1.2.** \{$\epsilon_0, \epsilon_1, \ldots, \epsilon_{N-1}$\} is an orthonormal basis for $L^2(\mathbb{Z}_N)$.

The proof of Proposition 1.2 is left as an exercise.

The orthonormal basis \{$\epsilon_0, \epsilon_1, \ldots, \epsilon_{N-1}$\} is the standard basis for $L^2(\mathbb{Z}_N)$. For another orthonormal basis for $L^2(\mathbb{Z}_N)$, we look at the signals in the following definition.

**Definition 1.3.** Let $e_0, e_1, \ldots, e_{N-1} \in L^2(\mathbb{Z}_N)$ be defined by
\[
e_m = \begin{pmatrix}
\epsilon_m(0) \\
\epsilon_m(1) \\
\vdots \\
\epsilon_m(N-1)
\end{pmatrix}, \quad m = 0, 1, \ldots, N-1,
\]