Chapter 10

Wavelet Transforms and Filter Banks

Let \( B = \{R_{2k}\varphi\}_{k=0}^{M-1} \cup \{R_{2k}\psi\}_{k=0}^{M-1} \) be a time-frequency localized basis for \( L^2(\mathbb{Z}_N) \), where \( \varphi \) is the mother wavelet and \( \psi \) is the father wavelet. For every signal \( z \) in \( L^2(\mathbb{Z}_N) \), we get, by the fact that \( B \) is an orthonormal basis for \( L^2(\mathbb{Z}_N) \) and (8.3),

\[
z = \sum_{k=0}^{M-1} (z, R_{2k}\varphi)R_{2k}\varphi + \sum_{k=0}^{M-1} (z, R_{2k}\psi)R_{2k}\psi. \tag{10.1}
\]

So, by (8.3),

\[
(z)_B = \begin{pmatrix}
(z, R_0\varphi) \\
(z, R_2\varphi) \\
\vdots \\
(z, R_{2M-2}\varphi) \\
(z, R_0\psi) \\
(z, R_2\psi) \\
\vdots \\
(z, R_{2M-2}\psi)
\end{pmatrix} = \begin{pmatrix}
(z * \varphi^*)(0) \\
(z * \varphi^*)(2) \\
\vdots \\
(z * \varphi^*)(2M-2) \\
(z * \psi^*)(0) \\
(z * \psi^*)(2) \\
\vdots \\
(z * \psi^*)(2M-2)
\end{pmatrix}, \quad z \in L^2(\mathbb{Z}_N). \tag{10.2}
\]

Let \( V_{\varphi,\psi} \) be the \( N \times N \) matrix defined by

\[
V_{\varphi,\psi} = (R_0\varphi| \cdots |R_{2M-2}\varphi|R_0\psi| \cdots |R_{2M-2}\psi).
\]

Then, by (10.1) and (10.2), we get

\[
z = (z)_S = V_{\varphi,\psi}(z)_B
\]

or

\[
(z)_B = V^{-1}_{\varphi,\psi}z, \quad z \in L^2(\mathbb{Z}_N).
\]

**Definition 10.1.** Let \( W_{\varphi,\psi} : L^2(\mathbb{Z}_N) \rightarrow L^2(\mathbb{Z}_N) \) be defined by

\[
W_{\varphi,\psi} = V^{-1}_{\varphi,\psi}.
\]
Then we call $W_{\varphi, \psi}$ the wavelet transform associated to the mother wavelet $\varphi$ and the father wavelet $\psi$.

**Remark 10.2.** The wavelet transform $W_{\varphi, \psi}$ is the change of basis matrix from the standard basis $S$ for $L^2(\mathbb{Z}_N)$ to the time-frequency localized basis $B$ generated by $\varphi$ and $\psi$.

**Remark 10.3.** Since $V_{\varphi, \psi}$ is a unitary matrix, it follows that

$$W_{\varphi, \psi} = V_{\varphi, \psi}^{-1} = V_{\varphi, \psi}^*,$$

where $V_{\varphi, \psi}^*$ is the adjoint of $V_{\varphi, \psi}$, i.e., the transpose of the conjugate of $V_{\varphi, \psi}$. So, an explicit formula for the wavelet transform $W_{\varphi, \psi}$ is available.

Using the explicit formula for the wavelet transform $W_{\varphi, \psi}$, we can compute the coordinates $(z)_B$ of every signal $z$ in $L^2(\mathbb{Z}_N)$ by means of the formula

$$(z)_B = W_{\varphi, \psi} z.$$

As has been pointed out, this computation may entail up to $N^2$ complex multiplications on a computer, and hence is not a feasible formula for the computation of $(z)_B$ for $z$ in $L^2(\mathbb{Z}_N)$. As a matter of fact, it is the formula (10.2) that people use to compute $(z)_B$ for every $z$ in $L^2(\mathbb{Z}_N)$. A look at (10.2) reveals that for every $z$ in $L^2(\mathbb{Z}_N)$, the components in $(z)_B$ are given by convolutions of $z$ with the involutions of the mother and father wavelets, and hence $(z)_B$ can be computed rapidly by the FFT. In order to exploit the structure of (10.2), we need a definition.

**Definition 10.4.** Let $N = 2M$, where $M$ is a positive integer. Then we define the linear operator $D : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_M)$ by

$$(Dz)(n) = z(2n), \quad n = 0, 1, \ldots, M - 1,$$

for all $z$ in $L^2(\mathbb{Z}_N)$.

What the linear operator $D$ does to a signal $z$ in $L^2(\mathbb{Z}_N)$ can best be seen by an example.

**Example 10.5.** Let $z$ be the signal in $L^2(\mathbb{Z}_8)$ defined by

$$z = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \\ 6 \\ 5 \\ 8 \\ 7 \end{pmatrix}.$$