Chapter 10
Wavelet Transforms and Filter Banks

Let \( B = \{ R_{2k} \varphi \}_{k=0}^{M-1} \cup \{ R_{2k} \psi \}_{k=0}^{M-1} \) be a time-frequency localized basis for \( L^2(\mathbb{Z}_N) \), where \( \varphi \) is the mother wavelet and \( \psi \) is the father wavelet. For every signal \( z \) in \( L^2(\mathbb{Z}_N) \), we get, by the fact that \( B \) is an orthonormal basis for \( L^2(\mathbb{Z}_N) \) and (8.3),

\[
z = \sum_{k=0}^{M-1} (z, R_{2k} \varphi) R_{2k} \varphi + \sum_{k=0}^{M-1} (z, R_{2k} \psi) R_{2k} \psi. \tag{10.1}
\]

So, by (8.3),

\[
(z)_B = \begin{pmatrix}
(z, R_0 \varphi) \\
(z, R_2 \varphi) \\
\vdots \\
(z, R_{2M-2} \varphi) \\
(z, R_0 \psi) \\
(z, R_2 \psi) \\
\vdots \\
(z, R_{2M-2} \psi)
\end{pmatrix} = \begin{pmatrix}
(z \ast \varphi^*)(0) \\
(z \ast \varphi^*)(2) \\
\vdots \\
(z \ast \varphi^*)(2M-2) \\
(z \ast \psi^*)(0) \\
(z \ast \psi^*)(2) \\
\vdots \\
(z \ast \psi^*)(2M-2)
\end{pmatrix}, \quad z \in L^2(\mathbb{Z}_N). \tag{10.2}
\]

Let \( V_{\varphi, \psi} \) be the \( N \times N \) matrix defined by

\[
V_{\varphi, \psi} = (R_0 \varphi| \cdots | R_{2M-2} \varphi| R_0 \psi| \cdots | R_{2M-2} \psi).
\]

Then, by (10.1) and (10.2), we get

\[
z = (z)_S = V_{\varphi, \psi}(z)_B
\]
or

\[
(z)_B = V_{\varphi, \psi}^{-1} z, \quad z \in L^2(\mathbb{Z}_N).
\]

**Definition 10.1.** Let \( W_{\varphi, \psi} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N) \) be defined by

\[
W_{\varphi, \psi} = V_{\varphi, \psi}^{-1}.
\]
Then we call \( W_{\varphi, \psi} \) the wavelet transform associated to the mother wavelet \( \varphi \) and the father wavelet \( \psi \).

**Remark 10.2.** The wavelet transform \( W_{\varphi, \psi} \) is the change of basis matrix from the standard basis \( S \) for \( L^2(\mathbb{Z}_N) \) to the time-frequency localized basis \( B \) generated by \( \varphi \) and \( \psi \).

**Remark 10.3.** Since \( V_{\varphi, \psi} \) is a unitary matrix, it follows that
\[
W_{\varphi, \psi} = V_{\varphi, \psi}^{-1} = V_{\varphi, \psi}^*,
\]
where \( V_{\varphi, \psi}^* \) is the adjoint of \( V_{\varphi, \psi} \), i.e., the transpose of the conjugate of \( V_{\varphi, \psi} \). So, an explicit formula for the wavelet transform \( W_{\varphi, \psi} \) is available.

Using the explicit formula for the wavelet transform \( W_{\varphi, \psi} \), we can compute the coordinates \((z)_B\) of every signal \( z \) in \( L^2(\mathbb{Z}_N) \) by means of the formula
\[
(z)_B = W_{\varphi, \psi} z.
\]
As has been pointed out, this computation may entail up to \( N^2 \) complex multiplications on a computer, and hence is not a feasible formula for the computation of \((z)_B\) for \( z \) in \( L^2(\mathbb{Z}_N) \). As a matter of fact, it is the formula (10.2) that people use to compute \((z)_B\) for every \( z \) in \( L^2(\mathbb{Z}_N) \). A look at (10.2) reveals that for every \( z \) in \( L^2(\mathbb{Z}_N) \), the components in \((z)_B\) are given by convolutions of \( z \) with the involutions of the mother and father wavelets, and hence \((z)_B\) can be computed rapidly by the FFT. In order to exploit the structure of (10.2), we need a definition.

**Definition 10.4.** Let \( N = 2M \), where \( M \) is a positive integer. Then we define the linear operator \( D : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_M) \) by
\[
(Dz)(n) = z(2n), \quad n = 0, 1, \ldots, M - 1,
\]
for all \( z \) in \( L^2(\mathbb{Z}_N) \).

What the linear operator \( D \) does to a signal \( z \) in \( L^2(\mathbb{Z}_N) \) can best be seen by an example.

**Example 10.5.** Let \( z \) be the signal in \( L^2(\mathbb{Z}_8) \) defined by
\[
z = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \\ 6 \\ 5 \\ 8 \\ 7 \end{pmatrix}.
\]