Finite-order Invariants for \((n, 2)\)-Torus Knots and the Curve \(Y^2 = X^3 + X^2\)

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**Abstract.** We describe the algebra of finite-order invariants on the set of all \((n, 2)\)-torus knots.

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**Keywords.** Finite type invariants, torus knots, polynomials.

This paper is an extended exposition of the talk [Ty] given by the first author. The authors thank S. Duzhin, A. Sossinsky for the interest to this work and S. Chmutov, O. Viro for useful discussions.

Consider the \(\mathbb{Q}\)-algebra \(V\) of Vassiliev finite-order knot invariants, see for example [B, CDL]. The algebra is filtered,

\[
V_0 \subset V_1 \subset \cdots \subset V_k \subset \cdots \subset V,
\]

the vector subspace \(V_k \subset V\) consists of knot invariants of order not greater than \(k\). We have \(V_k \cdot V_l \subset V_{k+l}\).

The subspace \(V_0\) is of dimension 1 and consists of invariants taking the same value on all knots. It is known that \(V_1 = V_0\,\), \(\dim V_2/V_1 = \dim V_3/V_2 = 1\). The generator in \(V_2/V_1\) is given by the knot invariant \(x\) of order 2 which takes value 0 on the trivial knot and value 8 on the trefoil. The generator in \(V_3/V_2\) is given by the knot invariant \(y\) of order 3 which takes value 0 on the trivial knot, takes value 24 on the trefoil, and takes value \(-24\) on its mirror image. Those conditions determine \(x\) and \(y\) uniquely, see for example [L].

It is known that the space \(V_k\) has finite dimension fast growing with \(k\), see for example [CD, D, Z].

By definition the algebra \(V\) is an algebra of certain special functions on the set \(K\) of all knots in \(\mathbb{R}^3\) considered up to isotopy.

Let \(T \subset K\) be the subset of toric knots of type \((n, 2)\), \(n = \pm 1, \pm 3, \ldots\). Here \((1, 2)\) and \((-1, 2)\) denote the trivial knot, \((3, 2)\) is the trefoil, \((-3, 2)\) its mirror image, and so on.

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Consider the algebra $A$ of functions on $T$, which is the restriction of $V$ to $T$, i.e., $A = V |_T$. The algebra $A$ is filtered,

$$A_0 \subset A_1 \subset \cdots \subset A_k \subset \cdots \subset A,$$

where $A_k = V_k |_T$ for any $k$. Our goal is to describe $A$.

Let $X \in A_2$ and $Y \in A_3$ be the image of $x$ and $y$, respectively, under the natural projection $V \to A$.

**Theorem.** The algebra $A$ is generated by $X$ and $Y$ and is isomorphic to the algebra $\mathbb{Q}[X,Y]/(X^3 + X^2 - Y^2)\mathbb{Q}[X,Y]$, where $(X^3 + X^2 - Y^2)\mathbb{Q}[X,Y] \subset \mathbb{Q}[X,Y]$ is the ideal generated by the polynomial $X^3 + X^2 - Y^2$. We have $\dim A_0 = 1$, $\dim A_1/A_0 = 0$, $\dim A_k/A_{k-1} = 1$ for $k > 1$. The generator in $A_{2l}/A_{2l-1}$ is given by $X^l$ and the generator in $A_{2l+1}/A_{2l}$ is given by $X^{l-1}Y$ for all $l > 0$.

**Proof.** Denote by $\mathbb{Z}_{\text{odd}}$ the set of all odd integers. For $n \in \mathbb{Z}_{\text{odd}}$ denote by $[n]$ the torus knot of type $(n,2)$.

An element $f \in A$ defines a function $\mathbb{Z}_{\text{odd}} \to \mathbb{Q}$, $n \mapsto f([n])$, and is uniquely determined by that function. Thus $A$ can be considered as an algebra of certain functions on $\mathbb{Z}_{\text{odd}}$.

**Lemma.**

- If $f : \mathbb{Z}_{\text{odd}} \to \mathbb{Q}$ belongs to $A_k$ for some $k$, then $f$ is a polynomial of degree not greater than $k$.
- If $f : \mathbb{Z}_{\text{odd}} \to \mathbb{Q}$ belongs to $A$, then $f(1) = f(-1)$. \hfill $\square$

The lemma is a direct corollary of definitions.

We have $X : \mathbb{Z}_{\text{odd}} \to \mathbb{Q}$, $n \mapsto n^2 - 1$, and $Y : \mathbb{Z}_{\text{odd}} \to \mathbb{Q}$, $n \mapsto n^3 - n$. This gives the relation $Y^2 = X^3 + X^2$.

It is easy to see that all polynomials $f : \mathbb{Z}_{\text{odd}} \to \mathbb{Q}$ with property $f(1) = f(-1)$ are linear combinations of monomials $X^l$ and $X^{l-1}Y$ of degree $2l$ and $2l+1$ respectively.

The theorem is proved. \hfill $\square$

**Remarks**

- After this paper had been written, S. Chmutov informed us about paper [T], where R. Trapp in particular shows that any element $f \in A$ is a polynomial function on $\mathbb{Z}_{\text{odd}}$ and $f$ can be written as a polynomial in $X$ and $Y$.
- S. Chmutov informed us that the shapes, similar to the shape of our curve $Y^2 = X^3 + X^2$, appeared in [W], where S. Willerton discusses the statistics of points $(x(k), y(k)) \in \mathbb{Q}^2$ for arbitrary knots $k$.
- According to our theorem the algebra $V |_T$ is isomorphic to the algebra of regular functions on the affine curve $Y^2 = X^3 + X^2$. One may wander what kind of algebraic schemes one obtains by considering restrictions of $V$ to other reasonable subsets of the set of all knots.