Chapter 2

Thermodynamic Formalism: Basic Notions

This chapter is an introduction to the classical thermodynamic formalism. Although everything is proven, we develop the theory in a pragmatic manner, only as much as needed for the following chapters. We start by introducing the notion of topological pressure, which includes the topological entropy as a special case. After describing some basic properties of the topological pressure, we establish its variational principle, and we show that there exist equilibrium measures for any expansive transformation. This also serves the purpose of presenting some of the central ideas of the theory without the more involved technicalities in later chapters. Moreover, we present the apparently not so well-known characterization of the topological pressure as a Carathéodory dimension. An elaboration of this approach will be very useful later, in particular to introduce the notion of nonadditive topological pressure in Chapter 4. For further developments of the thermodynamic formalism we refer to the books [108, 109, 149, 166, 195].

2.1 Topological pressure

We first introduce the notion of topological pressure in terms of separated sets. This is the most basic notion of the thermodynamic formalism, and it is a generalization of the notion of topological entropy. It also plays a fundamental role in the dimension theory and in the multifractal analysis of dynamical systems, as we substantially illustrate in later chapters.

Let $f : X \to X$ be a continuous transformation of a compact metric space $(X, d)$. For each $n \in \mathbb{N}$ we consider the distance $d_n$ in $X$ defined by

$$d_n(x, y) = \max \{ d(f^k(x), f^k(y)) : 0 \leq k \leq n - 1 \}. $$
Definition 2.1.1. Given $\varepsilon > 0$, a set $E \subset X$ is said to be $(n, \varepsilon)$-separated (with respect to $f$) if $d_n(x, y) > \varepsilon$ for every $x, y \in E$ with $x \neq y$.

We note that since the space $X$ is compact, each $(n, \varepsilon)$-separated set $E$ is finite. The notion of topological pressure can now be introduced as follows.

Definition 2.1.2. The topological pressure of a continuous function $\varphi : X \to \mathbb{R}$ (with respect to $f$) is defined by

$$P(\varphi) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{E} \sum_{x \in E} \exp \sum_{k=0}^{n-1} \varphi(f^k(x)),$$

where the supremum is taken over all $(n, \varepsilon)$-separated sets $E \subset X$.

We note that since the function

$$\varepsilon \mapsto \limsup_{n \to \infty} \frac{1}{n} \log \sup_{E} \sum_{x \in E} \exp \sum_{k=0}^{n-1} \varphi(f^k(x))$$

is nondecreasing, the limit in (2.1) when $\varepsilon \to 0$ is always well defined. The notion of topological pressure was introduced by Ruelle [164] for expansive transformations (see Definitions 2.4.2 and 2.4.5) and by Walters [194] in the general case.

The following example shows that the topological entropy is a particular case of the topological pressure. We recall that the topological entropy of $f$ is given by

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(d_n, \varepsilon),$$

where $N(d_n, \varepsilon)$ is the maximum number of points in $X$ at a $d_n$-distance at least $\varepsilon$.

Example 2.1.3. For the function $\varphi = 0$ and any $(n, \varepsilon)$-separated set $E$, we have

$$\sum_{x \in E} \exp \sum_{k=0}^{n-1} \varphi(f^k(x)) = \sum_{x \in E} 1 = \text{card } E.$$

Therefore,

$$\sup_{E} \sum_{x \in E} \exp \sum_{k=0}^{n-1} \varphi(f^k(x)) = N(d_n, \varepsilon),$$

which yields

$$P(0) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(d_n, \varepsilon) = h(f).$$

The original definition of topological entropy is due to Adler, Konheim and McAndrew [1]. The alternative definition in (2.2) was introduced independently by Bowen [36] and Dinaburg [48].