Chapter 1

Introduction and Terminology

This book deals with correspondence rules or rules of association. The fundamental idea is to associate a function of ordinary variables with an operator. While in later chapters we deal with arbitrary operators, in the main portion of the book we deal with the operators $X$ and $D$ where,

$$X = \begin{cases} x & \text{in the } x\text{ representation} \\ \frac{i}{d} & \text{in the Fourier representation} \end{cases}$$ 

$$D = \begin{cases} \frac{1}{i} & \text{in the } x\text{ representation} \\ p & \text{in the Fourier representation} \end{cases}$$

The fundamental relation between $X$ and $D$ is the commutator

$$[X, D] = XD - DX = i.$$ 

Depending on the field, these operators may be appropriately called position and spatial frequency, or position and momentum, and in time-frequency analysis they correspond to the time and frequency operators.

One of the basic reasons for considering these particular operators is that we can use them to evaluate expectation values of functions in the Fourier or $x$ representation without leaving the representation. In particular, suppose we have a function, $f(x)$, and a “state” function $\varphi(x)$; then

$$\int f(x) |\varphi(x)|^2 dx = \int \hat{\varphi}^*(p)f(X)\hat{\varphi}(p) dp$$

where $\hat{\varphi}(p)$ is the Fourier transform of $\varphi(x)$,

$$\hat{\varphi}(p) = \frac{1}{\sqrt{2\pi}} \int \varphi(x) e^{-i xp} dx.$$
Therefore, we say that $f(x)$ in the $x$ domain is associated with or represented by the operator $f(X)$ in the Fourier domain and we write

$$f(X) \leftrightarrow f(x) \quad (1.6)$$

where $\leftrightarrow$ indicates the association. Furthermore, suppose we have a function of $p$, $g(p)$, in the Fourier domain, then

$$\int g(p) |\hat{\varphi}(p)|^2 dp = \int \varphi^*(x) g(D)\varphi(x) dx \quad (1.7)$$

and we say that $g(p)$ in the Fourier domain is associated with $g(D)$ in the $x$ domain,

$$g(D) \leftrightarrow g(p). \quad (1.8)$$

The right hand side of Eq. (1.7) is sometimes called sandwiching because the operator, $g(D)$, is in between $\varphi^*(x)$ and $\varphi(x)$.

The basic advantage of Eq. (1.7) is that if we want to calculate the expectation of $g(p)$ as defined by the left hand side we do not have to first calculate $\hat{\varphi}(p)$. We can remain in the $x$ representation and use the right hand side to calculate it.

1.1 The Fundamental Issue

In the above discussion we had no difficulty in associating a function of one variable $f(x)$ or $g(p)$ with its corresponding operator. But what if we have a function of two variables, for example $xp^2$, then what will the association be? It could be, for example, $XD^2$, $DXD$, or $D^2X$, among others; all of these associations are proper in the sense that they reduce to $xp^2$ if we just think of the operators as ordinary variables. However, all these choices are different because $X$ and $D$ do not commute. Formulating such associations for a general function $a(x,p)$ is the fundamental aim. There have been many rules proposed, among them the Weyl, Standard, and Born-Jordan. In the next chapter we study the Weyl rule and subsequently other rules. In Chap. 4 we present a general method that handles all rules in a unified manner.

Furthermore, suppose we did have an association for the function $a(x,p)$ with an operator $A(X,D)$. Then, what is the generalization of Eqs. (1.4) and (1.7)? We will see that the proper generalization is that for a state function $\varphi(x)$ we have to introduce a joint function, $C(x,p)$ called the generalized distribution function which will allow us to write

$$\int \varphi^*(x) A(X,D)\varphi(x) dx = \iint a(x,p)C(x,p) dx dp \quad (1.9)$$

As we will see in Chap. 5 this is the generalization of Eqs. (1.4) and (1.7) and reduces to them when the symbol is a function of $x$ or $p$ only.