Chapter 10

The Full Version of Theorem 1.3

10.1 Statement of results

We use the same setup as in the previous chapter: \( L/E \) is a quadratic extension of totally real fields, \( \text{Gal}(L/E) = \langle 1, \varsigma \rangle \), \( \text{Gal}(L/E)^\wedge = \langle 1, \eta \rangle \), \( \epsilon \subset \mathcal{O}_L \) is an ideal, and
\[
\begin{align*}
    n & := [L: \mathbb{Q}], \\
    d & := d_{L/E}, \\
    \mathcal{D} & := \mathcal{D}_{L/E}, \\
    c_E & := \epsilon \cap \mathcal{O}_E.
\end{align*}
\]

We write \( L = E(\sqrt{\Delta}) \) for a totally positive \( \Delta \in E \). There is a distinguished type \( J_E = \{ \sigma \in \Sigma(L) : \sigma(\sqrt{\Delta}) < 0 \} \) for \( L/E \). We assume that the set \( \Sigma(L) \) is ordered so that for \( z \in \mathfrak{h}_{\Sigma(L)} \) we have
\[
    dz = dz_{\sigma_1} \wedge dz_{\sigma'_1} \wedge \cdots \wedge dz_{\sigma_{[E:Q]}} \wedge dz_{\sigma'_{[E:Q]}},
\]
where \( J'_E := J_E =: \{ \sigma'_1, \ldots, \sigma'_{[E:Q]} \} \).

Let \( \chi_E : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times \) be a Hecke character. Let \( \chi = \chi_E \circ N_{L/E} \) be the resulting character of \( L^\times \backslash \mathbb{A}_L^\times \) and let \( \chi_0 \) be the resulting character of \( K_0(c) \). Assume that the conductor of \( \chi \) divides \( \epsilon \) and that \( \chi_E(b_{\infty}) = b_{\infty}^{-k - 2m} \) for all \( b \in \mathbb{A}_E^\times \).

Let \( \kappa = (k, m) \) be a weight for \( E \) and let \( \hat{\kappa} = (\hat{k}, \hat{m}) \) be the corresponding weight for \( L \). We obtain a “diagonal” cycle \( Z_0 := Z_0(c_E) \) with a canonical flat section \( S \).

Let \( \theta^u \) be a Hecke character of \( L \), of finite order, with conductor \( b = f(\theta^u) \) and let \( w = m_\sigma + k_\sigma/2 \) (which is independent of the choice of \( \sigma \)). Define \( \theta = \theta^u \mid \cdot \mid^{w}_{\mathbb{A}_L} \).

We assume throughout this chapter that
\[
    \theta|_{\mathbb{A}_E^\times} = \chi_E\eta.
\]

For a proof that such a character \( \theta \) exists, see [Hid8, Lemma 2.1]. Set
\[
    c' = cb^2 \quad \text{and} \quad c'_E := c' \cap \mathcal{O}_E.
\]

The character \( \theta \) may be used to twist the cycle \( Z_0(c'_E) \) and its section \( S \) to obtain a cycle \( Z_\theta \) and flat section \( S_\theta \) of the local system \( \mathcal{L}^\vee(\hat{\kappa}, \chi_0) \), which gives rise to an intersection homology class \( [Z_\theta] \in \text{Im} H_n(X_n(c), \mathcal{L}^\vee(\hat{\kappa}, \chi_0)) \).

Let \( f \in S_\chi(K_0(\ell), \chi) \) be a simultaneous eigenform for all Hecke operators. Let \( J \subseteq \Sigma(L) \). Then the modular form \( f^{-t} \in S_{\chi}^{\text{coh}}(K_0(\ell), \chi) \) determines a differential form \( \omega_J(f^{-t}) \) which gives a “middle” intersection cohomology class \([\omega_J(f^{-t})] \in \oplus^n(X_0(\ell), L(\kappa, \chi_0))\). Write

\[
b' := \prod \{ p : p \mid d_{L/E} \text{ and } p \mid \mathfrak{f}(\theta) \}
\]

where \( \mathfrak{f}(\theta) \) denotes the conductor of \( \theta \). In Section 10.2 we will prove the following theorem:

**Theorem 10.1.** Assume as above that \( \theta|_{H_E} = \chi_E \eta \) and that \( J \subseteq \Sigma(L) \) is a type for \( L/E \). If \( f \) is a base change of a Hilbert cusp form \( g \) on \( E \) with nebentypus \( \chi_E \), then

\[
\langle [\omega_J(f^{-t})], [Z_\theta] \rangle_K
\]

is equal to

\[
\frac{c_1}{2[E:Q]} L_*^{*,d_{L/E}b\cap\mathcal{O}_E}(\text{Ad}(g) \otimes \eta, 1)L_{b'}(\text{As}(f \otimes \theta^{-1}), 1)
\]

where

\[
c_1 = \frac{C_3[K_0(\ell_E) : K_1(\ell_E)]N_{E/Q}(\ell_E)^2d_{E/Q}^\ell N_{E/Q}(d_{L/E})\text{Res}_{s=1}^\ell \zeta_{d_{E/K}\ell_E}(s)}{R_E 2^{k+2} 2^{-2}(\mathcal{O}_E/\ell_E)^\times}
\]

with \( C_3 \) defined as in Proposition 9.5 and where \( R_E \) is the regulator of \( E \).

If \( J \subseteq \Sigma(L) \) is not a type for \( L/E \) or \( f \) is not a base change of a Hilbert cusp form \( g \) on \( E \) with nebentypus \( \chi_E \), then

\[
\langle [\omega_J(f^{-t})], [Z_\theta] \rangle_K = 0.
\]

Here \( \{k\} := \sum_{\sigma \in \Sigma(E)} \kappa_\sigma \) for \( k \in \mathbb{Z}^{\Sigma(E)} \).

We now prepare some notation so that we may state the full version of Theorem 1.3. Let

\[
P_{\text{new}} : \oplus^n H_n(X_0(\ell), \mathcal{L}(\kappa, \chi_0)) \rightarrow \oplus^n H_n^{\text{new}}(X_0(\ell), \mathcal{L}(\kappa, \chi_0))
\]

be the canonical projection, where \( \oplus^n H_n^{\text{new}}(X_0(\ell), \mathcal{L}(\kappa, \chi_0)) \) is the subspace spanned by classes that are \( f \)-isotypical under the action of the Hecke algebra for some newform \( f \in S_{\chi}^{\text{new}}(K_0(\chi), \chi) \). Setting \( \gamma = P_{\text{new}}W_{\ell^{-1}}^{*-1}[Z_\theta] \) in our notation from Section 8.3, we have

\[
\Phi_{P_{\text{new}}W_{\ell^{-1}}^{*-1}[Z_\theta], \gamma_E} \in \oplus^n H_n^{\chi_E}(X_0(\ell), \mathcal{L}(\kappa, \chi_0)) \otimes S_{\chi}^{\kappa}(\mathcal{N}(\ell), \chi_E).
\]

by Theorem 8.4. Finally for \( g \in S_{\chi}^{\text{new}}(K_0(\ell), \chi) \) let

\[
A(g, J) := \frac{[\omega_J(g^{-t}), \omega_{\Sigma(L)-J}(g^{-t})]}{L_*(\text{Ad}(g), 1)} = \frac{T(g, J)(g, g)\mathcal{P}}{L_*(\text{Ad}(g), 1)}
\]

(10.1.1)