Chapter 6

Automorphic Vector Bundles and Local Systems

In this section we begin with the general theory of local systems, automorphic vector bundles, and automorphy factors. After describing the finite-dimensional representation theory of GL$_2$ we determine the explicit equations relating modular forms and differential forms.

In modern terminology, a Hilbert modular form $f$ may be identified with a section of a certain vector bundle $\mathcal{L}(\kappa, \chi_0)$ on $Y_0(\epsilon)$. It may also be identified with a differential form on $Y_0(\epsilon)$ with coefficients in $\mathcal{L}(\kappa, \chi_0)$. Each of these identifications takes some work. In the first place, the modular form $f$ takes values in the complex numbers, while the fibers of the vector bundle $\mathcal{L}(\kappa, \chi_0)$ are vector spaces of dimension $\geq 1$, so we need a way to convert a complex number into a vector in the appropriate vector space. This is accomplished by the mapping $P_z$ of equation (6.7.1). Secondly, the modular form $f$ is a function on the group $G(\mathbb{A})$ whereas we are looking for a section of a vector bundle on the Hilbert modular variety $Y_0(\epsilon)$. The translation between these two descriptions of $f$ involves an automorphy factor. These ideas are combined in Proposition 6.4 which gives the precise correspondence between Hilbert modular forms and sections of $\mathcal{L}(\kappa, \chi_0)$. In order to obtain a holomorphic differential $n$-form (with coefficients in $\mathcal{L}(\kappa, \chi_0)$) we essentially need to tensor with the top exterior power of the tangent bundle of $Y_0(\epsilon)$. This has the effect of raising the “weight” by 2. So the final result, stated in Proposition 6.5 (see [Hid3] Section 2 and Section 6) starts with a Hilbert modular form $f \in S^\text{coh}_\kappa (K_0(\epsilon), \chi)$ of weight $\kappa = (k, m)$ and central character $\chi : L^x \backslash A^x_L \to \mathbb{C}$ and constructs a differential form

$$\omega(f) \in \Omega^n (Y_0(\epsilon), \mathcal{L}(\kappa, \chi_0))$$

on $Y_0(\epsilon) = G(\mathbb{Q}) \backslash G(\mathbb{A})/K_0(\epsilon)K_\infty$, with coefficients in the local system $\mathcal{L}(\kappa, \chi_0)$ that corresponds to the representation $L(\kappa, \chi_0) = \text{Sym}^k(V^\vee) \otimes \text{det}^{-m} \otimes E(\chi_0^\vee)$ of $K_0(\epsilon)K_\infty \subset G(\mathbb{A})$. (The shift by 2 in the weight was already incorporated into the definition of the weight of the modular form $f$ so it appears in Proposition...
6.4 rather than Proposition 6.5.) In Section 6.11 it is explained that the group of connected components $G(\mathbb{R})/G(\mathbb{R})^0$ acts on $Y_0(\mathfrak{c})$ by complex conjugation on certain coordinates. The induced action on $H^*(Y_0(\mathfrak{c}), \mathcal{L}(\kappa, \chi_0))$ changes the Hodge $(p,q)$ type of the cohomology class.

6.1 Generalities on local systems

A rank $n$ topological complex vector bundle on a topological space $X$ is a surjective mapping $\pi : \mathcal{E} \to X$ together with an atlas $\mathcal{C} = \{(U, \phi_U)\}$ of local trivializations. Here $U \subset X$ is an open set (the collection of which are required to cover $X$) and

$$\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^n$$

is a homeomorphism that commutes with the projection to $U$. These local trivializations are required to have linear transition functions, that is, if $\phi_V : \pi^{-1}(V) \to V \times \mathbb{C}^n$ then on $U \cap V$ the resulting transition function

$$\phi_V \circ \phi_U^{-1} : (U \cap V) \times \mathbb{C}^n \to (U \cap V) \times \mathbb{C}^n$$

is given by $(x,v) \mapsto (x, h(x)v)$ where $h_{U,V} : U \cap V \to \operatorname{GL}_n(\mathbb{C})$. This linearity condition is equivalent to the existence of globally defined, continuous addition $\mathcal{E} \times_X \mathcal{E} \to \mathcal{E}$ and scalar multiplication $\mathbb{C} \times \mathcal{E} \to \mathcal{E}$ mappings with respect to which each of the local trivializations $\phi_U$ is linear on the fibers of $\pi$. If $X$ and $E$ are a smooth (resp. complex) manifolds and all the $\phi_U$ are smooth (resp. holomorphic) then $E$ is referred to as a smooth (resp. holomorphic) vector bundle.

Let $\pi : E \to X$ be a topological vector bundle. Suppose there exists an atlas $\mathcal{C} = \{(U, \phi_U)\}$ of local trivializations and a subgroup $\Gamma \subset \operatorname{GL}_n(\mathbb{C})$ such that the image of each of the transition mappings $h : U \cap V \to \operatorname{GL}_n(\mathbb{C})$ lies in $\Gamma$. Then we say that the structure group of $E$ can be reduced to $\Gamma$. For example, if $\pi : E \to X$ is a smooth vector bundle (over a smooth manifold $X$), then a choice of Hermitian metric on $E$ gives a reduction of $E$ to the unitary group $U(n) \subset \operatorname{GL}_n(\mathbb{C})$.

If the structure group of $E$ can be reduced to a discrete group $\Gamma \subset \operatorname{GL}_n(\mathbb{C})$ then we say that $E$ is a local coefficient system or, equivalently, that it has a discrete structure group. If $\pi : E \to X$ is a smooth vector bundle then a flat connection (meaning a smooth connection whose Riemannian curvature vanishes everywhere) on $E$ determines a reduction to a discrete structure group. Conversely, if the structure group of a smooth vector bundle $E$ is discrete, then $E$ admits a connection whose Riemann curvature vanishes everywhere. Thus, a local system on $X$ is the “same thing” as a smooth vector bundle with a flat connection.

Let $\pi : E \to X$ be a smooth (complex) vector bundle and let $\Omega^r(X, E)$ be the vector space of smooth differential $r$-forms with values in $E$. It is the space of smooth sections of the vector bundle $\wedge^r T^* X \otimes_{\mathbb{R}} E$. In order to define the exterior derivative $d\omega$ of a differential form $\omega \in \Omega^r(X, E)$ it is necessary to have a connection $\nabla_E$ on $E$. In this case, $dd\omega = \mathfrak{R} \wedge \omega$ where $\mathfrak{R} \in \Omega^2(X, \operatorname{Hom}(E,E))$ is the