A GRADIENT METHOD FOR APPROXIMATING SADDLE POINTS AND CONSTRAINED MAXIMA

Kenneth J. Arrow and Leonid Hurwicz

P-223

13 June 1951

The RAND Corporation
SANTA MONICA • CALIFORNIA
A GRADIENT METHOD FOR APPROXIMATING SADDLE POINTS AND CONSTRAINED MAXIMA

Kenneth J. Arrow and Leonid Hurwicz

1. Introduction.

In the following, X and Y will be vectors with components \(X_i, Y_j\). By \(X \geq 0\) will be meant \(X_i \geq 0\) for all \(i\). Let \(g(X), f_j(X)\) \((j = 1, \ldots, m)\) be functions with suitable differentiability properties, where \(f_j(X) \geq 0\) for all \(X\), and define

\[
F(X, Y) = g(X) + \sum_{j=1}^{m} Y_j \left(1 - \left[-f_j(X) \right]^{1+\gamma} \right).
\]

Let \((\bar{X}, \bar{Y})\) be a saddle-point of (1) subject to the conditions \(X \geq 0, Y \geq 0\); assume it unique in \(X\). The function \(F(X, \bar{Y})\) attains its maximum for variation in \(X\) subject to the condition \(X \geq 0\) at the point \(X = \bar{X}\). Since \(F\) is a maximum for variation in each component \(X_i\) separately, it follows that

\[
\frac{\partial F}{\partial X_i} \leq 0 \quad \text{for all } i, \quad \text{and}
\]

\[
\bar{X}_i = 0 \quad \text{if } \frac{\partial F}{\partial X_i} < 0.
\]

We will refer to those subscripts for which (3) holds as corner indices and the remainder as interior indices. Let \(X^1\) be the vector of components of \(X\) with corner indices, and \(X^2\) the vector of interior components. Since \(F\) is also a maximum for variations in \(X^2\) alone (holding \(X^1\) at 0 and \(Y\) at \(\bar{Y}\)), and the first-order terms vanish by (2) and (3), it follows, under the usual differentiability assumptions, that the matrix,

\[
\frac{\partial^2 F}{\partial X_i \partial X_j} \quad \text{is negative semi-definite},
\]

- 1 -