Chapter 4

Geometry and Dynamics of Automorphisms of $\mathbb{P}^2_\mathbb{C}$

In this chapter we study and describe the geometry, dynamics and algebraic classification of the elements in $\text{PSL}(3, \mathbb{C})$, extending Goldman’s classification for the elements in $\text{PU}(2, 1) \subset \text{PSL}(3, \mathbb{C})$. Just as in that case, and more generally for the isometries of manifolds of negative curvature, the automorphisms of $\mathbb{P}^2_\mathbb{C}$ can also be classified into the three types of elliptic, parabolic and loxodromic (or hyperbolic) elements, according to their geometry and dynamics. This classification can be also done algebraically, in terms of their trace.

It turns out that elliptic and parabolic elements in $\text{PSL}(3, \mathbb{C})$ are all conjugate to elliptic and parabolic elements in $\text{PU}(2, 1)$. Also, the loxodromic elements in $\text{PU}(2, 1)$ are loxodromic as elements in $\text{PSL}(3, \mathbb{C})$, but in this latter group we get new types of loxodromic elements that cannot exist in $\text{PU}(2, 1)$. In this chapter we study and describe in detail each of these types of automorphisms of $\mathbb{P}^2_\mathbb{C}$.

Notice that when we look at subgroups of $\text{PU}(2, 1)$ we have the stringent condition of preserving the corresponding quadratic form, and therefore one has an invariant ball. Then the elliptic elements are those having a fixed point in the interior of the ball, parabolics have a fixed point in the boundary of the ball and loxodromic elements have two fixed points in the boundary. Yet, when we think of automorphisms of $\mathbb{P}^2_\mathbb{C}$, this type of classification makes no sense, since in general there is not an invariant ball or sphere.

In $\mathbb{P}^2_\mathbb{C}$ we must think globally. Each such an automorphism $\gamma$ has a lifting to $\text{SL}(3, \mathbb{C})$ with three eigenvalues (possibly not all distinct). Each eigenvector gives rise to a fixed point of $\gamma$. All these fixed points, and their local properties, must be taken into account for a classification of the elements in $\text{PSL}(3, \mathbb{C})$.

The material in this chapter is essentially contained in [160]. The first section gives a qualitative overview of the classification problem we address in this chapter. This somehow serves as an introduction to the topic. In the following sections...
we make precise the notions of elliptic, parabolic and loxodromic elements in PSL(3, \mathbb{C}), as well as the various subclasses of maps one has in each type.

We carefully describe the geometry and dynamics in each case. We determine in each case the corresponding Kulkarni limit set, the equicontinuity region and the maximal region of discontinuity. We also give an algebraic characterisation of the various types of transformations in terms of the trace of their liftings to SL(3, \mathbb{C}).

4.1 A qualitative view of the classification problem

We consider an element \( g \in \text{PSL}(3, \mathbb{C}) \) and all its iterates \( g^n := g \circ g^{n-1} \), for all \( n \in \mathbb{Z} \) (with \( g_1 := g, g_0 := \text{Id} \) and \( g^{-n} := (g^{-1})^n \)). In other words, we are considering the cyclic group generated by \( g \). The element \( g \) is represented by a matrix \( \tilde{g} \) in GL(3, \mathbb{C}), unique up to multiplication by nonzero complex numbers.

Such a matrix \( \tilde{g} \) has three eigenvalues, say \( \lambda_1, \lambda_2, \lambda_3 \), which may or may not be equal, and if they are distinct, they may or may not have equal norms: These facts make big differences in their geometry and dynamics, as we will see in the sequel. These, together with the corresponding Jordan canonical form of \( \tilde{g} \), yield to the geometric and dynamical characterisations of the elements in PSL(3, \mathbb{C}) that we give in this chapter. Yet, in the case of PSL(3, \mathbb{C}) we give also an algebraic classification in terms of the trace, and this looks hard to do in higher dimensions.

Notice also that what really matters are the ratios amongst the \( \lambda_i \), since multiplication of a matrix by a scalar multiplies all its eigenvalues by that same scalar. Recall also that each eigenvalue determines a one-dimensional space of eigenvectors in \( \mathbb{C}^3 \), so its projectivisation fixes the corresponding point in \( \mathbb{P}^2_\mathbb{C} \). Distinct eigenvalues give rise to distinct fixed points in \( \mathbb{P}^2_\mathbb{C} \). Also, every two points in \( \mathbb{P}^2_\mathbb{C} \) determine a unique projective line; if the two points are fixed by \( g \), then the corresponding line is \( g \)-invariant.

Let us use this information to have a closer look at the dynamics of \( g \) by considering a lifting \( \tilde{g} \in \text{SL}(3, \mathbb{C}) \) and looking at its Jordan canonical form. One can check that this must be of one of the following three types:

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}, \quad \text{where} \quad \lambda_3 = (\lambda_1 \lambda_2)^{-1},
\]

\[
\begin{pmatrix}
\lambda_1 & 1 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}, \quad \text{where} \quad \lambda_3 = (\lambda_1)^{-2},
\]

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}.
\]

Let us see what happens in each case. In the first case, the images of \( e_1, e_2, e_3 \) are fixed by the corresponding map in \( \mathbb{P}^2_\mathbb{C} \); for simplicity we denote the corresponding