9. Global Investigations

9.1 The Plücker formulae

Now that we have investigated the local properties of plane curves and their singular points from various viewpoints, in the previous paragraph, we next want to derive global assertions about plane complex projective algebraic curves. Above all, we shall calculate global invariants of such curves from the local invariants of their singular points which we have investigated earlier. The global invariants in question are the order, class and genus of curves. Formulae for the order and class were first presented by J. Plücker in 1834, and later generalised by M. Noether 1875 and 1883. We want to deal with these formulae in the present section. Then in the next section we shall derive the formula for the genus, which in special cases goes back to Riemann 1857 and Clebsch 1864, and in its general form to M. Noether 1874 and Weierstrass.

We have already defined the order of a plane curve $C$ without multiple components, at the beginning of our analytic-algebraic investigations of curves in 2.3 and 5.1, as the degree of the polynomial describing the curve $C$. The order is the simplest conceivable invariant, and its significance was already seen by Newton. It is an invariant, not of an abstract curve, but of the curve $C$ in the projective plane. The local analogue of this invariant is the multiplicity of a curve germ, which we have defined in 5.3.

As soon as the idea of duality emerged in the development of projective geometry, one could associate further invariants with a curve $C$, by considering the known invariants, but for the dual curve $C'$ of $C$. In 6.2 we have defined the dual curve $C'$ as the set of tangents of $C$, where these tangents are regarded as points in the dual projective plane. The order of this dual curve $C'$ is then a new invariant of $C$, the class of $C$. It is important that this invariant likewise depends on the embedding in the projective plane, and that to define it one needs the geometry of the plane, since one must be able to view tangents as lines in the plane and points in the dual plane.

It is clear how one must define a corresponding local invariant for the branch of a plane algebraic curve: the class of such an irreducible branch is the multiplicity of the corresponding point on the

dual curve. For a reducible curve germ the class is the sum of the classes of the branches.

Example:

Let $C$ be an irreducible cubic with a cusp. In suitable coordinates $(x_0, x_1, x_2)$, $C$ has the equation:

$$x_0 x_1^2 + x_2^3 = 0.$$ 

The point $(x'_0, x'_1, x'_2)$ of the dual curve which corresponds to the point $(x_0, x_1, x_2)$ of $C$ satisfies

$$(x'_0, x'_1, x'_2) = (x_1^2, 2x_0 x_1, 3x_2^2).$$ 

By eliminating $x_0$, $x_1$, $x_2$ from this and the equation of $C$ one obtains the equation

$$27x'_0 x'_1^2 - 4x'_2^3 = 0$$

for $C'$. Thus the dual curve $C'$ is again a cubic with a cusp. The cusp $(1,0,0)$ of $C$ corresponds to the inflection point $(0,1,0)$ of $C'$, and the inflection point $(0,1,0)$ of $C$ corresponds to the cusp $(1,0,0)$ of $C'$.

It is very pleasant to see the relation between the form of a curve and its dual, for suitably chosen examples, illustrated by pictures of the corresponding real curves.

In order to construct the dual of a given curve $C$ without a lot of calculation, one can describe the duality which associates points of the plane $E$ with lines of the dual plane $E'$, and lines of $E$ with points of $E'$, in terms of the pole-polar relationship for a circle: each point $p$ is associated with its polar in the circle, each line with its pole.