Metaplectic Equivalence of the Hierarchical Twisted Laplacian

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Abstract. We use a metaplectic operator to prove that the hierarchical twisted Laplacian $L_m$ is unitarily equivalent to the tensor product of the one-dimensional Hermite operator and the identity operator on $L^2(\mathbb{R}^{m+1})$, and we use this unitary equivalence to show that $L_m$ is globally hypoelliptic in the Schwartz space and in the Gelfand–Shilov spaces.

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1. Introduction

For all $x \in \mathbb{R}$ and all $v = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^m$, let

$$z = x + is(v),$$

where $s(v) = v_1 + v_2 + \cdots + v_m$. Then we let $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ be linear partial differential operators on $\mathbb{R}^{m+1}$ defined by

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i \sum_{l=1}^{m} \frac{\partial}{\partial v_l} + \frac{i}{2} \left( 1 - \frac{1}{m} \right) s(v),$$

and

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i \sum_{l=1}^{m} \frac{\partial}{\partial v_l} + \frac{i}{2} \left( 1 - \frac{1}{m} \right) s(v).$$

Then the hierarchical twisted Laplacian $L_m$ is defined on $\mathbb{R}^{m+1}$ by

$$L_m = -\frac{1}{2}(Z\bar{Z} + \bar{Z}Z),$$

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where

\[ Z = \frac{\partial}{\partial z} + \frac{1}{2} \bar{z} \quad \text{and} \quad \bar{Z} = \frac{\partial}{\partial \bar{z}} - \frac{1}{2} z. \]

By an easy calculation, we see that \( L_m \) is the linear partial differential operator on \( \mathbb{R}^{m+1} \) given by

\[
L_m = -\left( \frac{\partial^2}{\partial x^2} + \sum_{j,l=1}^{m} \frac{\partial^2}{\partial v_j \partial v_l} \right) + \frac{1}{4} \left( x^2 + \frac{s(v)^2}{m^2} \right) - i \left( \frac{s(v)}{m} \frac{\partial}{\partial x} - x \sum_{j=1}^{m} \frac{\partial}{\partial v_j} \right).
\]

In the case when \( m = 1 \), \( L_1 \) is a linear partial differential operator on \( \mathbb{R}^2 \) and has the form

\[
L_1 = -\Delta + \frac{1}{4} (x^2 + v_1^2) - i \left( v_1 \frac{\partial}{\partial x} - x \frac{\partial}{\partial v_1} \right),
\]

which is the ordinary twisted Laplacian and we denote it by \( L \). It is a perturbation of the Hermite operator by a rotation operator.

The twisted Laplacian \( L \) comes up as the quantum-mechanical Hamiltonian of the motion of an electron in the infinite two-dimensional plane under the influence of a constant magnetic field perpendicular to the plane. The eigenvalues of the system are known as Landau levels and the corresponding eigenfunctions are the Wigner transforms of Hermite functions. The twisted Laplacian has been studied extensively in, e.g., [4, 5, 6, 7, 11, 12, 13, 14, 16, 17, 18].

The twisted Laplacian \( L \) can in fact be obtained from the sub-Laplacian on the one-dimensional Heisenberg group \( \mathbb{C} \times \mathbb{R} \) by taking the Fourier transform with respect to the center, and is hence a linear partial differential operator on \( \mathbb{R}^2 \). Higher-dimensional twisted Laplacians on \( \mathbb{R}^{2n} \) can be obtained similarly using the \( n \)-dimensional Heisenberg group \( \mathbb{C}^n \times \mathbb{R} \). Therefore the twisted Laplacian is defined on even-dimensional Euclidean spaces. The hierarchical twisted Laplacian \( L_m \) on \( \mathbb{R}^{m+1} \) can be seen as a twisted Laplacian on \( \mathbb{R}^n \), where the dimension \( n \) can now be even or odd with \( n > 1 \).

The ordinary twisted Laplacian is well known to be elliptic, but not globally elliptic in the sense of Shubin defined in Section 25 of [10]. By explicit formulas of the heat kernel and the Green function of \( L \), it is shown in [18] by Wong that \( L \) is globally hypoelliptic in the Schwartz space \( \mathcal{S}(\mathbb{R}^2) \) and in [4] by Dasgupta and Wong that \( L \) is globally hypoelliptic in Gelfand–Shilov spaces. In [7], the global hypoellipticity of the twisted Laplacian is recaptured using the fact that \( L \) is unitarily equivalent to a tensor product of the ordinary Hermite operator and the identity operator.

The hierarchical twisted Laplacian \( L_m \) can be written in the form

\[
L_m = \left( D_x - \frac{1}{2m} s(v) \right)^2 + \left( \sum_{j=1}^{m} D_{v_j} + \frac{1}{2} x \right)^2.
\]