Chapter 20

The QR-Factorization Based Method

In this chapter we present a method for the inversion of block matrices with given quasiseparable representations without any restriction on the matrix except its invertibility. It is based on a special representation of a block invertible matrix $A$ in the form

$$A = VUR,$$  \hspace{1cm} (20.1)

where $V$ is a block lower triangular unitary matrix and $U$ is a block upper triangular unitary matrix, with nonsquare blocks, and $R$ is a block upper triangular matrix with square invertible blocks on the main diagonal. This is a form of the QR factorization of the matrix $A$ in which the unitary $Q$-factor is written in a special form.

The matrices $V$, $U$, $R$ are given by their quasiseparable generators, which are computed via quasiseparable generators of the original matrix $A$. Using this representation we find the solution of the system of linear algebraic equations $Ax = y$ as $x = R^{-1}U^*V^*y$. As a result, we obtain a linear complexity algorithm to find the solution $x$.

In the first step of the method we compute the factorization $A = VT$, where $V$ is a block lower triangular unitary matrix and $T$ is a block upper triangular matrix. In general, these matrices have rectangular blocks on the main diagonal. In order to obtain matrices which are convenient for inversion, we compute for the matrix $T$ the factorization $T = UR$, where $U$ is a block upper triangular unitary matrix and $R$ is a block upper triangular matrix with square invertible blocks on the main diagonal. Below we present the description of both steps with the detailed justification.

§20.1 Factorization of triangular matrices

We derive here factorizations which are valid for any block triangular matrices with given quasiseparable generators.
Lemma 20.1. Let $A = \{A_{ij}\}_{i,j=1}^{N}$ be a block lower triangular matrix with entries of sizes $m_i \times n_j$ and lower quasiseparable generators $p(i)$ $(i = 2, \ldots, N)$, $q(j)$ $(j = 1, \ldots, N - 1)$, $a(k)$ $(k = 2, \ldots, N - 1)$ and diagonal entries $d(k)$ $(k = 1, \ldots, N)$. Using these generators and the diagonal entries define matrices

$$A_1 = \begin{pmatrix} d(1) & 0 \\ q(1) & 0 \end{pmatrix}, \quad A_k = \begin{pmatrix} p(k) & d(k) \\ a(k) & q(k) \end{pmatrix}, \quad k = 2, \ldots, N-1, \quad A_N = \begin{pmatrix} p(N) & d(N) \\ 0 & 0 \end{pmatrix} \tag{20.2}$$

and then set

$$\bar{A}_1 = \text{diag}\{A_1, I_{\gamma_1}\}, \quad \bar{A}_k = \text{diag}\{I_{\eta_k}, A_k, I_{\gamma_k}\}, \quad k = 2, \ldots, N-1, \quad \bar{A}_N = \text{diag}\{I_{\eta_N}, A_N\}, \tag{20.3}$$

where $\eta_k = \sum_{i=1}^{k-1} m_i$, $\gamma_k = \sum_{i=k+1}^{N} n_i$.

Then

$$A = \bar{A}_N \bar{A}_{N-1} \cdots \bar{A}_1. \tag{20.4}$$

Proof. Let us prove by induction the validity of the relations

$$\bar{A}_k \cdots \bar{A}_1 = \begin{pmatrix} A(1 : k, 1 : k) & 0 \\ 0 & Q_k \\ 0 & 0 \end{pmatrix}, \quad k = 1, \ldots, N-1, \tag{20.5}$$

where the matrices $Q_k$ are given by (5.1).

For $k = 1$ (20.5) is obvious. Suppose (20.5) holds for $k$ with $1 \leq k \leq N - 2$. Then

$$\bar{A}_{k+1} \bar{A}_k \cdots \bar{A}_1 = \begin{pmatrix} I_{\eta_k+1} & 0 & 0 & 0 \\ 0 & p(k+1) & d(k+1) & 0 \\ 0 & a(k+1) & q(k+1) & 0 \\ 0 & 0 & 0 & I_{\gamma_k+1} \end{pmatrix} \begin{pmatrix} A(1 : k, 1 : k) & 0 & 0 \\ 0 & Q_k & 0 \\ 0 & 0 & I_{\eta_{k+1}} \end{pmatrix} \begin{pmatrix} A(1 : k+1, 1 : k+1) & 0 & 0 \\ 0 & p(k+1)Q_k & d(k+1) \\ 0 & a(k+1)Q_k & q(k+1) \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Using the equality (5.11) we get

$$\begin{pmatrix} A(1 : k, 1 : k) & 0 \\ p(k+1)Q_k & d(k+1) \end{pmatrix} = A(1 : k+1, 1 : k+1)$$

and thus using (5.3) we conclude that

$$\bar{A}_{k+1} \bar{A}_k \cdots \bar{A}_1 = \begin{pmatrix} A(1 : k+1, 1 : k+1) & 0 \\ 0 & Q_{k+1} \end{pmatrix}.$$