Chapter 3

Matrices in Diagonal Plus Semiseparable Form

Here we consider diagonal plus semiseparable representations of matrices. This is a direct generalization of diagonal plus separable representations studied in Chapter 1. Note that every matrix may be represented in the diagonal plus semiseparable form. The problem is to obtain such a representation with minimal orders. This may be treated as the problem of completing strictly lower triangular and strictly upper triangular parts of a matrix to matrices with minimal ranks, since it will be proved that minimal orders of the generators are equal to the ranks of those minimal completions. Thus one can apply results of Chapter 2 to determine diagonal plus semiseparable representation of a matrix. An algorithm for finding minimal generators of a semiseparable representation of a given matrix is presented.

§3.1 Definitions

Let \( A = \{A_{ij}\}_{i,j=1}^N \) be a matrix with block entries of sizes \( m_i \times n_j \). Denote by \( \tilde{A}_L = \{A_{ij}\}_{1 \leq j < i \leq N} \) the strictly lower triangular part of \( A \). We treat \( \tilde{A}_L \) as a given lower triangular part of a partially specified \( (N-1) \times (N-1) \) block matrix \( B = \{B_{ij}\}_{i=2,j=1}^{N-1,N-1} \) with \( B_{ij} = A_{ij} \) for \( 1 \leq j < i \leq N \). The minimal completion rank \( \hat{r}_L \) of \( \tilde{A}_L \) is called the lower semiseparable order of the matrix \( A \).

Similarly denote by \( \tilde{A}_U = \{A_{ij}\}_{1 \leq i < j \leq N} \) the strictly upper triangular part of \( A \). We treat \( \tilde{A}_U \) as a given upper triangular part of a partially specified \( (N-1) \times (N-1) \) block matrix \( C = \{C_{ij}\}_{i=1,j=2}^{N-1,N-1} \) with \( B_{ij} = A_{ij} \) for \( 1 \leq i < j \leq N \). The minimal completion rank \( \hat{r}_U \) of \( \tilde{A}_U \) is called the upper semiseparable order of the matrix \( A \).

We say also that \( A \) has semiseparable order \( (\hat{r}_L, \hat{r}_U) \).

Let \( S = \{S_{ij}\}_{i,j=1}^{N} \) be a matrix with block entries \( S_{ij} \) of sizes \( m_i \times n_j \) and...
with zero diagonal. Assume that the entries of $S$ are represented in the form

$$S_{ij} = \begin{cases} 
p(i)q(j), & 1 \leq j < i \leq N, \\
0, & 1 \leq i = j \leq N, \\
g(i)h(j), & 1 \leq i < j \leq N. 
\end{cases} \quad (3.1)$$

Here $p(i)$ ($i = 2, \ldots, N$), $q(j)$ ($j = 1, \ldots, N - 1$) are matrices of sizes $m_i \times r_L$, $r_L \times n_j$, respectively; $g(i)$ ($i = 1, \ldots, N - 1$), $h(j)$ ($j = 2, \ldots, N$) are matrices of sizes $m_i \times r_U$, $r_U \times n_j$, respectively. The representation of the matrix $S$ in the form (3.1) is called a \textit{semiseparable representation}. The elements $p(i)$ ($i = 2, \ldots, N$), $q(j)$ ($j = 1, \ldots, N - 1$) and $g(i)$ ($i = 1, \ldots, N - 1$), $h(j)$ ($j = 2, \ldots, N$) are called \textit{semiseparable generators} of the matrix $S$. The numbers $r_L$ and $r_U$ are called the \textit{orders} of these generators.

The representation (3.1) means that if we introduce the matrices

$$P = \text{col}(p(i))_{i=2}^N, \quad Q = \text{row}(q(j))_{j=1}^{N-1}$$

of sizes $(\sum_{i=2}^N m_i) \times r_L, r_L \times (\sum_{j=1}^{N-1} n_j)$ and the matrices

$$G = \text{col}(g(i))_{i=1}^{N-1}, \quad H = \text{row}(h(j))_{j=2}^N$$

of sizes $(\sum_{i=1}^{N-1} m_i) \times r_U, r_U \times (\sum_{j=2}^N n_j)$ and we define the $(N - 1) \times (N - 1)$ block matrices $S_L = PQ$ and $S_U = GH$ of ranks $r_L$ and $r_U$ at most, then the relations

$$\text{trils}(S) = \text{tril}(S_L), \quad \text{trius}(S) = \text{triu}(S_U),$$

hold. Here tril($X$), triu($X$) denote the lower triangular and upper triangular parts of a matrix $X$ and trils($S$), trius($S$) denote strictly lower triangular and strictly upper triangular parts of the matrix $S$. In other words, the strictly lower triangular and the strictly upper triangular parts of the matrix $S$ may be completed to some matrices $S_L$ and $S_U$ with the ranks not greater than $r_L$ and $r_U$, respectively.

Let $A$ be a matrix with block entries $A_{ij}$ of sizes $m_i \times n_j$ represented as a sum $A = D + S$ of a block diagonal matrix $D = \text{diag}(d(i))_{i=1}^N$ with the entries $d(i)$ of sizes $m_i \times n_i$ and a matrix $S$ with semiseparable representation (3.1). The entries of $A$ are specified as follows:

$$A_{ij} = \begin{cases} 
p(i)q(j), & 1 \leq j < i \leq N, \\
d(i), & 1 \leq i = j \leq N, \\
g(i)h(j), & 1 \leq i < j \leq N. 
\end{cases} \quad (3.2)$$

The representation of the matrix $A$ in the form (3.2) is called \textit{diagonal plus semiseparable representation}.

The elements $p(i)$ ($i = 2, \ldots, N$), $q(j)$ ($j = 1, \ldots, N - 1$) and $g(i)$ ($i = 1, \ldots, N - 1$), $h(j)$ ($j = 2, \ldots, N$) are called lower and upper semiseparable generators of the matrix $A$. The most interesting case for us is when for a given matrix $A$ the orders $r_L$ and $r_U$ are minimal. Lower and upper semiseparable generators of $A$ with minimal orders are called \textit{minimal semiseparable generators}.