Chapter 7

Unitary Matrices with Quasiseparable Representations

In this chapter we study in detail the quasiseparable representations of unitary matrices. We show that for unitary matrices the quasiseparable representations are closely connected with factorization representations of a matrix as a product of elementary unitary matrices.

In the first section we present with the proof the well-known results on Givens rotations and QR factorizations of matrices. In the second section we derive relations between rank numbers of unitary matrices. In the third section we study factorization representations of unitary matrices and their connections with quasiseparable representations. In the fourth section we consider a special case of unitary Hessenberg matrices. In the last section we study special quasiseparable representations of matrices for which computations may be performed with a lower complexity.

§7.1 QR and related factorizations of matrices

Let $A$ be an $m \times n$ matrix. Then $A$ may be represented in the form

$$A = Q \cdot R$$

with an $m \times m$ unitary matrix $Q$ and an $m \times n$ matrix $R = (R_{ij})$ such that

$$R_{ij} = 0 \quad \text{for} \quad i > j.$$  

The factorization (7.1) is called the QR factorization of the matrix $A$. To determine the factors $Q$ and $R$ one can proceed as follows.

For a two-dimensional complex vector $x = \begin{pmatrix} a \\ b \end{pmatrix}$ there is a complex Givens
rotation matrix, i.e., a $2 \times 2$ unitary matrix $G$, such that

$$G x = \begin{pmatrix} r \\ 0 \end{pmatrix}$$

with some complex number $r$. The matrix $G$ and the number $r$ may be determined by the formulas

$$G = \begin{pmatrix} c & s \\ -s^* & c^* \end{pmatrix}, \quad r = \sqrt{|a|^2 + |b|^2},$$

where

$$c = \frac{a^*}{r}, \quad s = \frac{b^*}{r}$$

for $x \neq 0$ and $c = 1, s = 0$ for $x = 0$.

At first we determine a complex Givens rotation matrix $G_{m-1}$ from the condition

$$G_{m-1}A(m-1:m, 1) = \begin{pmatrix} r \\ 0 \end{pmatrix}.$$

Define the $m \times m$ unitary matrix $\tilde{G}_{m-1} = I_{m-2} \oplus G_{m-1}$. Then the matrix $A_1 = \tilde{G}_{m-1}A$ has a zero entry in the $(m, 1)$ position. Next we determine a complex Givens rotation matrix $G_{m-2}$ from the condition

$$G_{m-2}A(m-2:m-1, 1) = \begin{pmatrix} r' \\ 0 \end{pmatrix}$$

and we define the $m \times m$ unitary matrix $\tilde{G}_{m-2} = I_{m-3} \oplus G_{m-2} \oplus I_1$. The matrix $A_2 = \tilde{G}_{m-2}A_1$ has zero entries in the $(m-1, 1)$ and $(m, 1)$ positions. We proceed in the same way with the first columns of the matrices $A, A_1, A_2, \ldots, A_{m-2}$ and obtain the matrix

$$A^{(1)} := A_{m-1} = \tilde{G}_1 \cdots \tilde{G}_{m-1}A$$

with all the entries zero, except for the first one in the first column:

$$A^{(1)}(2:m, 1) = 0.$$

Here $G^{(1)} = \tilde{G}_1 \cdots \tilde{G}_{m-1}$ is an $m \times m$ unitary matrix.

Next we apply the same procedure to the second column of the matrix $A^{(1)}$. We determine a complex Givens rotation matrix $G^{(1)}_{m-1}$ from the condition

$$G^{(1)}_{m-1}A^{(1)}(m-1:m, 2) = \begin{pmatrix} r \\ 0 \end{pmatrix}.$$