INTERTWININGS AND HYPERINVARIANT SUBSPACES

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Let $E$ be a Banach space, and let $L(E)$ denote the Banach algebra of all bounded linear operators on $E$. A nontrivial hyperinvariant subspace for an operator $A$ in $L(E)$ is a nonzero, proper, (closed) subspace of $E$ which is invariant under any operator in $[A]'$, the commutant of $A$.

The purpose of this note is to present a general principle which leads to the existence of nontrivial hyperinvariant subspaces for certain operators. In particular we recapture many of the results asserting the existence of hyperinvariant subspaces for operators satisfying growth conditions either global or local ([3], [4], [6], [7], [10], [12]).

Our principle is based on the construction of intertwinings between a given operator and an operator of multiplication by $z$ on a certain Banach algebra of functions. We already used it in a special case to make certain reductions to "good cases" in applications of the S. Brown technique [2]. An additional reason which motivates this approach is its close connection with local spectral theory.

The paper is organized as follows. In Section 1 we present the general principle which boils down to the well-known fact that if two operators have disjoint spectra then these two operators cannot be nontrivially intertwined. Section 2 contains some preliminary material on regular Banach algebras and intertwinings. The main results are given in Section 3. In Section 4 we show the connection of our approach with local spectral theory. Section 5 contains the applications to operators satisfying growth conditions. In Section 6 we recall how this principle was used in connection with the S. Brown technique.
1. THE GENERAL PRINCIPLE

We begin by recalling the following well-known fact.

PROPOSITION 1.1. ([9]). Let $E$ and $F$ be two Banach spaces, let $A$ and $B$ be operators on $E$ and $F$ respectively, and let $T:E \to F$ be a bounded linear operator intertwining $B$ with $A$ (i.e. $AT=TB$). If the spectra of $A$ and $B$ are disjoint then $T=0$.

The following proposition contains the key idea of our way of connecting intertwinings and hyperinvariant subspaces.

PROPOSITION 1.2. Let $E$ be a Banach space, let $A \in L(E)$, let $U_1 \in L(B_1)$, $U_2 \in L(B_2)$ where $B_1$ and $B_2$ are Banach spaces, and let $T_1 \in L(B_1, E)$, $T_2 \in L(B_2, E^*)$ such that:

1. $AT_1 = T_1 U_1$,
2. $A^* T_2 = T_2 U_2$,
3. $\sigma (U_1) \cap \sigma (U_2) = \emptyset$.

Then $A$ has a nontrivial hyperinvariant subspace.

PROOF. (The reader is advised to follow the intertwinings with the help of the usual commutative diagrams.) We will denote by $j$ the canonical embedding of $E$ into its bidual $E^{**}$. Recall that $A^{**} = jA$ and that $j(A)$ is weak* dense in $A^{**}$. Let $B$ be any operator commuting with $A$. Then $A(BT_1) = (BT_1)U_1$. This equality combined with the intertwinings $A^{**} = jA$ and $U_2^{*}T_2 = T_2A^{**}$ (this latter one being the dual of (2)) yields the following equality

$$U_2^{*}(T_2^{*}jBT_1) = (T_2^{*}jBT_1)U_1.$$ 

Since $\sigma (U_1) \cap \sigma (U_2^{*}) = \emptyset$ we deduce from Proposition 1.1 that $T_2^{*}jBT_1 = 0$, that is $\text{Ran}(BT_1) \subseteq \text{Ker}(T_2^{*}j)$ for every $B$ in $\{A\}'$. Since $T_2^{*}$ is continuous when $E^{**}$ and $B_2^{*}$ are equipped with their respective weak* topologies the operator $T_2^{*}j$ is nonzero (otherwise the weak* density of $j(E)$ in $E^{**}$ would imply $T_2^{*} = 0$ and hence $T_2 = 0$).

Thus $\text{Ker}(T_2^{*}j)$ is a proper subspace of $E$ and the closed linear span of $\text{Ran}(BT_1)$ when $B$ runs over $\{A\}'$ is a nontrivial