SMOOTHED PREDICTOR-CORRECTOR METHODS FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. Special predictor-corrector methods employing residue smoothing for solving semidiscrete partial differential equations are analysed. By the technique of residue smoothing the stability condition is relaxed to such an extent that the (explicit) PC methods can be applied with time steps prescribed by accuracy considerations rather than by stability considerations. The additional computational effort involved by the explicit smoothing technique used here is rather low when compared with its stabilizing effect. However, the overall accuracy may be decreased. This paper investigates the effect of residue smoothing on the accuracy.

1. INTRODUCTION

We consider the numerical solution of the initial-value problem for systems of (nonlinear) ordinary differential equations (ODEs) of the form

\[ \frac{d^{\nu} y(t)}{dt^{\nu}} = f(t, y(t)), \quad \nu = 1, 2, \tag{1} \]

which arise when time-dependent partial differential equations (PDEs) are semidiscretized in space. We shall assume that the Jacobian matrix \( \frac{\partial f}{\partial y} \) has eigenvalues located in a negative interval \([-R, 0)\). In dealing with such systems of ODEs, we have to take into account that the spectral radius \( R \) is usually extremely large. Therefore, we need an integration method with a large real stability boundary. Restricting our considerations to the class of linear multistep methods, we are led to implicit methods and as a con-
sequence we are faced with the problem of solving in each integration step the implicit equation
\[ y - b_0 r^\nu f(t_{n+1}, y) = \Sigma_n, \quad \Sigma_n := \sum_{i=1}^{k} \left[-a_i y_{n+1-i} + b_i r^\nu f(t_{n+1-i}, y_{n+1-i})\right], \quad (2) \]
where \( r \) denotes the integration step, \( y_{n+1-i} \) presents an approximation to the exact solution \( y(t) \) at \( t = t_{n+1-i} \) and the coefficients \( a_i \) and \( b_i \) define the linear multistep method. The solution of this equation provides a numerical approximation to \( y(t) \) at \( t = t_{n+1} \). In practice, equation (2) is only approximately solved and this approximate solution is accepted as the numerical approximation \( y_{n+1} \) to \( y(t) \) at \( t = t_{n+1} \).

In this paper, we pursue our earlier investigation of the smoothed generalized predictor-corrector (SGPC) methods for finding approximate solutions to equation (2). These methods were proposed in [2] for the first-order case \( \nu = 1 \), and we here we will extend them to the case \( \nu = 2 \). Furthermore, we analyse the effect of the relaxation parameters occurring in the SGPC scheme on the accuracy and the stability of the SGPC method for more general problems than the model problem considered in [2].

2. \hspace{1cm} SGPC METHODS

We consider SGPC methods of the form
\[ y^{(j)} = y^{(j-1)} - r^{(j)} S[y^{(j-1)} - b_0 r^\nu f(t_{n+1}, y^{(j-1)}) - \Sigma_n], \quad j = 1, 2, \ldots, m, \quad (3) \]
where the \( r^{(j)} \) are relaxation parameters, \( S \) is a smoothing matrix, and \( y^{(0)} \) is an initial approximation which will be assumed to be obtained by an (explicit) linear \( k \)-step method. Evidently, if this method converges for \( m \to \infty \), then it will converge to the solution of (2). Notice that the conventional PC method is obtained if we set \( r^{(j)} S = I \). Following the terminology used in PC methods, we shall call (3) an SGPC method in P(ESC)\textsuperscript{m}E mode.

The method (3) may be considered as a two-level iteration scheme for approximating the solution of equation (2). In [2] the more general multi-level SGPC methods have been considered, but, for the sake of transparency, we shall confine our considerations to the two-level version (3). However, from an implementational point of view, the two-level version is sometimes less attractive, so that in our numerical experiments multi-level versions of (3) are used (see Section 7.1).