GALERKIN BOUNDARY ELEMENT METHOD WITH SINGLE LAYER POTENTIAL

Michio Sakakihara

Abstract. Galerkin method for an integral equation on a boundary $\partial \Omega$ of a bounded domain in $\mathbb{R}^2$, arising from a Dirichlet boundary value problem for an elliptic partial differential equation is considered in this paper. By using a single layer potential corresponding to the problem we obtain an integral equation on the boundary. The main result of the paper is that the integral equation has a unique solution in the Sobolev space $H^{1/2}(\partial \Omega)$. We also give its $H^1(\Omega)$-error estimate.

1. INTRODUCTION

As a numerical method for solving a Dirichlet boundary value problem such as

\[-\Delta u + u = 0 \text{ in } \Omega,\]

\[u = g \text{ on } \partial \Omega,\]  

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with the $C^2$-boundary $\partial \Omega$ the boundary element method is suitable to obtain the discretized version and solve. When we formulate an integral equation on the boundary with the single layer potential representation of the function which satisfies the equation (1.1), we are led to Fredholm integral equation of the first kind. In this case it is important to prove that the integral equation has a unique solution in an appropriate Sobolev space. A discussion of the integral equation arising from Laplace equation was presented by Nedelec and Planchard [4]. They proved that a bilinear form arising from a Dirichlet problem for Laplace
equation in $\mathbb{R}^3$ is $H^{-1/2}(\partial \Omega)$-elliptic. Then a variational problem on the boundary corresponding to the problem has a unique solution. For the case in $\mathbb{R}^2$ Le Roux [7] obtained the same results. The results for Laplace equation were also presented by Okamoto [5] using a different method from Nedelec and Planchard's. Applications of the boundary element method to equations such as (1.1) appear in the formulations of numerical methods for partial differential equations, such as steady convective diffusion problems [3], Laplace transformed equations of transient diffusion equations, semi-discrete equation in time for transient diffusion equations [9], and convective diffusion problems with first order reaction [6]. Furthermore in some linearizations with quasi-Newton methods for mildly non-linear partial differential equations [8], we also find some examples. Such examples will be shown in the final section.

Let us now consider the boundary element method for the problem (1.1), (1.2). It is shown that the integral equation on the boundary corresponding to the problem (1.1), (1.2) has a unique solution in $H^{-1/2}(\partial \Omega)$, that when we discretize the integral equation by Galerkin method the Galerkin solution converge to the exact solution and that we obtain $H^1(\Omega)$-error estimate. To this end the author uses a different theory from Nedelec and Planchard. To prove unique existence of the solution we shall apply the results presented by Babuška [1].

2. INTEGRAL EQUATION

The single layer potential representation corresponding to the equation (1.1) is expressed as

$$U(x) = \frac{1}{2\pi} \int_{\partial \Omega} K_0(|x-y|) p(y) ds(y),$$

(2.1)

where $x = (x_1,x_2)$, $y = (y_1,y_2)$ and $|x-y|$ is the distance between the points $x$ and $y$, $K_0$ denotes the second kind modified Bessel function which is a fundamental solution for the equation (1.1), $p$ is a density function defined on the boundary and $s$ denotes the arc length of the boundary. Here we denote $x$ the coordinate of the point in $\Omega$. It is obvious that

$$-\Delta U(x) + U(x) = 0.$$ 

(2.2)

The integral equation on the boundary for the problem (1.1-2):