ERROR ESTIMATE FOR A QUADRATURE FORMULA FOR $H^2$ FUNCTIONS

ALEXANDER G. RAMM

Kansas State University, Mathematics Department, Manhattan, KS, U.S.A.

Abstract

Let $f \in H^2$, where $H^2$ is the Hardy space on the unit disc. Let $-1 < x_1 < x_2 < \ldots < x_n < 1$ be fixed given numbers. Consider $\sup_{\|f\| \leq 1} \left| \int_{-1}^{1} f(t) dt - \sum_{j=1}^{n} a_j f(x_j) \right| := \epsilon(a_1, \ldots, a_n) := \epsilon(a)$. Here $\| \cdot \|$ is the norm in $H^2$, $a_j = \text{const}$, $a = (a_1, \ldots, a_n)$. The quantity $\epsilon_n := \min_a \epsilon(a)$ is computed.

Introduction

Let $f \in H^2$, $H^2$ is the Hardy space on the unit disc. Let $-1 < x_1 < \cdots < x_n < 1$ be fixed given numbers. Define $\epsilon(a) := \epsilon(a_1, \ldots, a_n) := \sup_{\|f\| \leq 1} \left| \int_{-1}^{1} f(t) dt - \sum_{j=1}^{n} a_j f(x_j) \right|$, where $a_j = \text{const}$, $\| f \|$ is the norm in $H^2$. We are interested in computing $\epsilon_n := \min_a \epsilon(a)$ where minimum is taken over all complex numbers $a_j$. In the literature there are several papers on the related question of estimating $\gamma_n := \min_{a,x} \epsilon(a)$, where $x_j$ are allowed to vary [1]-[3]. It is proved in [1] that $\frac{3}{2} \exp(-3\sqrt{2}\sqrt{n}) \leq \gamma_n \leq 11 \exp\left(-\frac{\sqrt{n}}{2\sqrt{2}}\right)$, and in [3] that $\lim_{n \to \infty} \gamma_n^{1/\sqrt{n}} = \exp\left(-\frac{\pi}{\sqrt{2}}\right)$. The problem of estimating or calculating of $\epsilon_n$ when $x_1, \ldots, x_n$ are fixed was formulated by C. Michelli at this conference. In this short note a method is given for calculating $\epsilon_n$.

Calculation of $\epsilon_n$

Note that $\int_{-1}^{1} f(t) dt = \int_C f(t) h(t) dt$, where $\int_C := \int_{|t|=1}, h(t) := \left\{ \begin{array}{ll} -\frac{1}{2} & \text{if } |t| = 1, Im t > 0 \\ \frac{1}{2} & \text{if } |t| = 1, Im t < 0 \end{array} \right.$, $f(x_j) = (2\pi i)^{-1} \int_C f(t)(t-x_j)^{-1} dt$. Thus

$$\epsilon(a) = \sup_{\|f\| \leq 1} \left| \int_C dt f(t) \left[ h(t) - (2\pi i)^{-1} \sum_{j=1}^{n} a_j (t-x_j)^{-1} \right] \right|$$

$$= \| h_+ - (2\pi i)^{-1} \sum_{j=1}^{n} a_j (t-x_j)^{-1} \| L^2(C) \quad (1)$$
where
\[ h(t) := \sum_{m<0} h_m t^m, \quad h_m := \frac{1}{2\pi} \int_0^{2\pi} h(t) \exp(-imt) dt, \]  
(2)
so that
\[ h(t) = \sum_{p=0}^{\infty} \frac{2}{-i} \frac{t^{-2p+1}}{2p+1} := \sum_{p=0}^{\infty} h_{2p+1} t^{-2p+1}, h_{2p+1} = 0, h_{2p} = \frac{2}{-i} \frac{1}{2p+1}. \]  
(3)
In (1) we have used the known formula for the norm of the linear functional on \( H^2 \) [4]. From (1), (3), and Parseval's equality it follows that
\[
\epsilon(a) = \| h(t) - \sum_{p=0}^{\infty} \left( \sum_{j=1}^{n} b_j x_j^p \right) t^{-(p+1)} \|_{L^2(C)} = \left( \sum_{p=0}^{\infty} h_p - \sum_{j=1}^{n} b_j x_j^p \right)^{1/2}.
\]  
(4)
Therefore
\[
\min_a \epsilon(a) := \epsilon_n = G(H, f_1, \ldots, f_n)/G(f_1, \ldots, f_n).
\]  
(5)
Here
\[
G(f_1, \ldots, f_n) := \det(f_m, f_j), 1 \leq m, j \leq n
\]  
(6)
is the Gramian, \((f, g)\) is the inner product in \( \ell^2 \), where \( \ell^2 \) is the space of sequences \( f_m = (f_m^0, f_m^1, \ldots) \) with the inner product \((f_m, f_j) := \sum_{p=0}^{\infty} f_{mp} \overline{f_{jp}}\), the bar stands for complex conjugate,
\[
H = (h_0, h_1, \ldots)
\]  
(7)
with \( h_j \) defined in (3), and the known formula (5) for the distance from \( H \) to the subspace spanned by the vectors
\[
f_j = (1, x_j, x_j^2, \ldots), 1 \leq j \leq n
\]  
(8)
was used [5, §1.13]. One has
\[
(f_m, f_j) = \sum_{p=0}^{\infty} x_m^p x_j^p = (1 - x_m x_j)^{-1} := b_{mj}.
\]  
(9)
Also
\[
(H, f_j) = \frac{2}{-i} \sum_{p=0}^{\infty} \frac{x_j^{2p}}{2p+1} = 2ix_j^{-1} \sum_{p=0}^{\infty} \frac{x_j^{2p+1}}{2p+1} = ix_j^{-1} \ln \frac{1 + x_j}{1 - x_j} := \phi(x_j)
\]  
\[
(H, H) = 4 \sum_{p=0}^{\infty} (2p+1)^{-2} = \frac{\pi^2}{2}.
\]  
(10)