Chapter 1

Two good counting algorithms

Counting problems that can be solved exactly in polynomial time are few and far between. Here are two classical examples whose solution makes elegant use of linear algebra. Both algorithms predate the now commonplace distinction between polynomial and exponential time, which is often credited (with justification) to Edmonds in the mid 1960s; indeed our first example dates back over 150 years!

1.1 Spanning trees

Basic graph-theoretic terminology will be assumed. Let $G = (V, E)$ be a finite undirected graph with vertex set $V$ and edge set $E$. For convenience we identify the vertex set $V$ with the first $n$ natural numbers $[n] = \{0, 1, \ldots, n - 1\}$. The adjacency matrix $A$ of $G$ is the $n \times n$ symmetric matrix whose $ij$’th entry is 1 if $\{i, j\} \in E$, and 0 otherwise. Assume $G$ is connected. A spanning tree in $G$ is a maximum (edge) cardinality cycle-free subgraph (equivalently, a minimum cardinality connected subgraph that includes all vertices). Any spanning tree has $n - 1$ edges.

Theorem 1.1 (Kirchhoff). Let $G = (V, E)$ be a connected, loop-free, undirected graph on $n$ vertices, $A$ its adjacency matrix and $D = \text{diag}(d_0, \ldots, d_{n-1})$ the diagonal matrix with the degrees of the vertices of $G$ in its main diagonal. Then, for any $i$, $0 \leq i \leq n - 1$,

$$\# \text{ spanning trees of } G = \det(D - A)_{ii},$$

where $(D - A)_{ii}$ is the $(n - 1) \times (n - 1)$ principal submatrix of $D - A$ resulting from deleting the $i$’th row and $i$’th column.

Since the determinant of a matrix may be be computed in time $O(n^3)$ by Gaussian elimination, Theorem 1.1 immediately implies a polynomial-time algorithm for counting spanning trees in an undirected graph.
Chapter 1: Two good counting algorithms

Example 1.2. Figure 1.1 shows a graph $G$ with its associated “Laplacian” $D - A$ and principal minor $(D - A)_{11}$. Note that $\det(D - A)_{11} = 3$ in agreement with Theorem 1.1.

Remark 1.3. The theorem holds for unconnected graphs $G$, as well, because then the matrix $D - A$ associated with $G$ is singular. To see this, observe that the rows and columns of a connected graph add up to 0 and, similarly, those of any submatrix corresponding to a connected component add up to 0. Now choose vertex $i$ and a connected component $C$ such that $i \not\in C$. Then, the columns of $(D - A)_{ii}$ that correspond to $C$ are linearly dependent, and $(D - A)_{ii}$ is singular.

Our proof of Theorem 1.1 follows closely the treatment of van Lint and Wilson [65], and relies on the following expansion for the determinant, the proof of which is deferred.

Lemma 1.4 (Binet-Cauchy). Let $A$ be an $(r \times m)$- and $B$ an $(m \times r)$-matrix. Then

$$\det AB = \sum_{S \subseteq [m], |S| = r} \det A_{*S} \det B_{S*},$$

where $A_{*S}$ is the square submatrix of $A$ resulting from deleting all columns of $A$ whose index is not in $S$, while, similarly, $B_{S*}$ is the square submatrix of $B$ resulting from $B$ by deleting those rows not in $S$.

Remark 1.5. Typically, $r$ is smaller than $m$. However, the lemma is also true for $r > m$. Then the sum on the right is empty and thus 0. But also $AB$ is singular, since rank $AB \leq$ rank $A \leq m < r$.

Let $H$ be a directed graph on $n$ vertices with $m$ edges. Then the incidence matrix of $H$ is the $(n \times m)$-matrix $N = (\nu_{ve})$ where

$$\nu_{ve} = \begin{cases} +1, & \text{if vertex } v \text{ is the head of edge } e; \\ -1, & \text{if } v \text{ is the tail of } e; \\ 0, & \text{otherwise.} \end{cases}$$

The weakly connected components of $H$ are the connected components of the underlying undirected graph, i.e., the graph obtained from $H$ by ignoring the orientations of edges.