Chapter 6

Volume of a convex body

We arrive at one of the most important applications of the Markov chain Monte Carlo method: the estimation of the volume of a convex body. For a convex body $K$ in low dimensional Euclidean space, say two or three dimensions, it is not too difficult to estimate the volume of $K$ within reasonable relative error using a direct Monte Carlo approach. Depending on how $K$ is presented, it may even be possible to find the volume exactly without too much difficulty. In this chapter, therefore, we imagine the dimension $n$ of the space to be large, and certainly greater than 3.

There are two related problems:

- sample uniformly at random a point from the convex body $K$;
- estimate the volume $\text{vol}_n K$ of $K$.

We will first look at the problem of random sampling in $K$. Since volume is the limit of a sum, it is not surprising, in the light of examples contained in previous chapters, that the second problem can be reduced to the first. We shall look first at the problem of random sampling in $K$; the reduction of volume estimation to sampling will be covered at the end of the chapter.

The convex body is given as an oracle which, for a point $x \in \mathbb{R}^n$, tells whether or not $x \in K$ (see Figure 6.1). This oracle model subsumes several possible conventions for describing inputs. For example, in the case of a convex polytope defined by a set of linear inequalities it is of course easy to implement the oracle. A convex polytope presented as the convex hull of its vertices it is a little harder, but it can still be done, by linear programming. In some applications, the assumption of an exact oracle that accurately decides whether $x \in K$ may be unrealistic. In an implementation we would almost certainly be using arithmetic with bounded precision, and we could not always know for sure whether were in or out. In fact, it is possible to relax the definition of oracle to incorporate some fuzziness at the boundary of $K$ without loosing much algorithmically. One of the many simplifications we shall make in this chapter is to assume exact arithmetic and an exact oracle. For a much fuller picture, refer to Kannan, Lovász and Simonovits [39].
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Figure 6.1: Oracle for $K$.

Figure 6.2: Sampling by “direct” Monte Carlo.

The first thing to be noticed in this endeavour is that some intuitively appealing approaches do not work very well. Let us consider a conventional application of the Monte Carlo method to the problem. Say we shrink a box $C$ around $K$ as tightly as possible (see Figure 6.2), sample a point $x$ uniformly at random from $C$, and return $x$ if $x \in K$; otherwise repeat the sampling if $x \notin K$. This simple idea works well in low dimension, but not in high dimension, where the volume ratio $\frac{\text{vol}_n K}{\text{vol}_n C}$ can be exponentially small. This phenomenon may be illustrated by a very simple example. Let $K = B_n(0,1)$ be the unit ball, and $C = [-1,1]^n$ the smallest enclosing cube. In this instance the ratio in question may be calculated exactly, and is $\frac{\text{vol}_n K}{\text{vol}_n C} = 2\pi^{n/2}/(2^n n \Gamma(n/2))$, which decays rapidly with $n$.\footnote{The Gamma function extends the factorial function to non-integer values. When $n$ is even, $\Gamma(n/2) = (n/2 - 1)!$, so it is easy to see that the ratio $\frac{\text{vol}_n K}{\text{vol}_n C}$ tends to 0 exponentially fast.} In the light of this observation, it seems that a random walk through $K$ may provide a better alternative.

Dyer, Frieze and Kannan [23] were the first to propose a suitable random walk for sampling random points in a convex body $K$ and prove that its mixing time scales as a polynomial in the dimension $n$. As a consequence, they obtained the first FPRAS for the volume of a convex body. Needless to say, this result was a major breakthrough in the field of randomised algorithms. Their approach was to divide $K$ into a $n$-dimensional grid of small cubes, with transitions available between cubes sharing a facet (i.e., an $(n-1)$-dimensional face). This proposal imposes a preferred coordinate system on $K$ leading to some technical complications. Here, instead, we use the coordinate-free “ball walk” of Lovász and Simonovits [44].